Department of Physics, Princeton University

# Graduate Preliminary Examination
# Part II

Friday, May 10, 2013
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number. No calculators are permitted.

On the front of each booklet write out and sign the statement: "This paper represents my own work in accordance with University regulations."

# Section A. Quantum Mechanics

1. **Hyperfine Structure**

The hyperfine structure of the $n = 1$ level of the hydrogen atom arises from a coupling between the electron spin $\mathbf{S}_e$ and the proton spin $\mathbf{S}_p$ with Hamiltonian

$$H = A\mathbf{S}_e \cdot \mathbf{S}_p \ ,$$

where $A$ is a positive constant. Use the convention where the spin operators are dimensionless. The kinetic energy and Coulomb interaction do not lift the spin degeneracies and may be ignored in this problem.

(a) What are the energies and degeneracies of the hyperfine levels in the absence of a magnetic field?

A uniform magnetic field $\mathbf{B}$ is turned on for a period of time. Assume that the field is constant for $0 < t < T$ and zero for all other times.

(b) To a good approximation you can ignore the coupling of the proton spin to the magnetic field, compared to that of the electron spin. Briefly explain why this is true.

(c) Assume the atom was in the hyperfine state with total spin zero for $t < 0$. What is the probability that it remains in this state for $t > T$?

2. **Delta function potential**

Consider a particle of mass $m$ moving nonrelativistically in one dimension subject to an attractive delta-function potential $V(x) = -V_0\delta(x)$, with $V_0 > 0$.

(a) What are the energy and the normalized wavefunction of this particle's ground state?

(b) The particle is perturbed by a weak additional time-dependent potential

$$V_1(x, t) = Fx \cos(\omega t) \ .$$

What is the transition rate from the ground state to the continuum? [It might be helpful to confine the particle in a large "box" $|x| < L$ and then take the limit $L \to \infty$.]

3. **Spins on a square**

Four spin-$S$ spins are located at the corners of a square and interact antiferromagnetically ($J > 0$). Use the convention where the spin operators are dimensionless. The Hamiltonian is

$$H = J(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_4 + \mathbf{S}_4 \cdot \mathbf{S}_1) \ .$$

(a) What are a complete set of good quantum numbers that can be used to fully classify all of the eigenstates of $H$?

(b) For spin-1/2 give the eigenenergy and the degeneracy of each energy level.

(c) For general spin $S$, what are the energy, degeneracy, and quantum numbers of the ground state?

# Section B. Thermodynamics and Statistical Mechanics

1. **Carnot Engine**

A Carnot engine uses $n$ moles of an ideal gas as its working substance. The absolute temperatures of its hot and cold reservoirs are denoted by $T_H$ and $T_C$ respectively. The net work performed by the engine in one cycle of operation is $W$. An investigator is asked to check the values of the reservoir temperatures, but unfortunately she is not provided with a thermometer. However she is able to measure W, and also the following volumes:

$V_{H1}$ = volume of working substance when first contacted with the hot reservoir
$V_{H2}$ = volume of working substance after extracting heat from the hot reservoir
$V_{C1}$ = volume of working substance when first contacted with the cold reservoir
$V_{C2}$ = volume of working substance after giving up heat to the cold reservoir

Derive expressions for the unknown temperatures $T_H$ and $T_C$ in terms of $n$, $W$, the ratios of the above volumes, the molar gas constant $R$ and the ratio $\gamma$ of the constant pressure and constant volume specific heats for the gas.

2. **Graphene**

Graphene is a two-dimensional sheet of carbon atoms. Both electronic and phononic degrees of freedom contribute to the low temperature specific heat per unit area. The energies of the electron states near the Fermi energy are

$$\epsilon_\pm(\vec{p}) = \epsilon_0 \pm v_F p,$$

where $\vec{p} = (p_x, p_y)$ is the the momentum of the electron and $p \equiv |\vec{p}|$. (There are two energy bands, $\epsilon_+(\vec{p}) \geq \epsilon_0$ and $\epsilon_-(\vec{p}) \leq \epsilon_0$, which are degenerate at $p = 0$.) These states have a fourfold degeneracy at each value of the momentum: the usual two-fold spin degeneracy is doubled by an additional valley index.

(a) If the Fermi energy $\epsilon_F$ is $\epsilon_0 + v_F p_F$ with $p_F > 0$, what is the leading behavior of the electronic specific heat as $T \to 0$?

(b) What is the leading low-temperature electronic specific heat when $p_F = 0$?

The next calculation is independent of parts a) and b) above.

Recently freely suspended Graphene sheets have been studied. These have an unusual phonon spectrum. In addition to the two dimensional longitudinal and transverse waves with frequencies $\omega = v_L q, v_T q$ at wavenumber of magnitude $q = |\vec{q}|$, there is an extra low-frequency mode with $\omega = Kq^2$ with the atomic displacements are normal to the sheet.

(c) Obtain the leading behavior of the phonon contribution to the specific heat as $T \to 0$.

You may express your answers in terms of the numerical constants

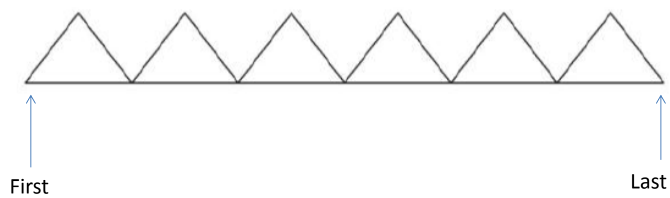
$$C_n^\pm = \int_0^\infty dx \frac{x^n}{e^x \pm 1}$$

3. **Frustrated and Unfrustrated Magnetism**

Consider the triangular chain shown below. We will study the antiferromagnetic Ising model on this lattice defined by the Hamiltonian

$$H = J \sum_{\langle ij \rangle} S_i S_j$$

where the spins $S_i$ live on the sites of the lattice and take values $\pm 1$, the exchange is antiferromagnetic ($J > 0$) and the sum runs over all nearest neighbor pairs $\langle ij \rangle$ on the lattice (vertices joined by links in the figure).




(a) Consider a system consisting of a single triangle at $T = 0$. How many ground states (minimum energy configurations) does it have?

(b) Consider a system of $N$ triangles at $T = 0$. How many ground states does the chain have, supposing free boundary conditions?

(c) Calculate the correlation $\langle S_{\text{first}} S_{\text{last}} \rangle$ averaged over the ground states for the first and last spins on the bottom row of the chain with $N$ triangles.

(d) How would the answers to the above three questions change if the exchange were made ferromagnetic instead, i.e. $J < 0$?

(Hint: In parts (b) and (c) fairly simple counting arguments will give exact results.)