Prelims Solutions

Problem M12M2

Valentin Skoutnev

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T_M of the mass M is its rotational KE about its center of mass (which is constant due to the motor) plus its translational KE which is $\frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)$. T_m of mass m is $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$.

$$x = rcos(\theta) + acos(\Omega t),\ y = rsin(\theta) + asin(\Omega t)$$

$$\dot{x} = \dot{r}cos(\theta) - r\dot{\theta}sin(\theta) - a\Omega sin(\Omega t),\ \dot{y} = \dot{r}sin(\theta) + r\dot{\theta}cos(\theta) + a\Omega cos(\Omega t)$$

So

$$T_m = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + ma\dot{r}\Omega sin(\theta - \Omega t) + mr\dot{\theta}\Omega cos(\theta - \Omega t)$$

Lastly, the only potential energy is in the spring so $U = \frac{1}{2}kr^2$.

Lagrangians are indifferent to constants so

$$L = T_M + T_m - U = \frac{1}{2}(M+m)(\dot{r}^2 + r^2\dot{\theta}^2) + ma\dot{r}\Omega sin(\theta - \Omega t) + mr\dot{\theta}\Omega cos(\theta - \Omega t) - \frac{1}{2}kr^2$$

E-L equations for r give:

$$\frac{\partial L}{\partial r} = (M+m)r\dot{\theta}^2 + ma\dot{\theta}\Omega cos(\theta - \Omega t) - kr = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = (M+m)\ddot{r} + ma\Omega cos(\theta - \Omega t)(\dot{\theta} - \Omega)$$

$$\rightarrow (M+m)r\dot{\theta}^2 - kr = (M+m)\ddot{r} - ma\Omega^2 cos(\theta - \Omega t)$$

E-L equations for θ give:

$$\frac{\partial L}{\partial \theta} = ma\dot{r}\Omega cos(\theta - \Omega t) - mar\dot{\theta}sin(\theta - \Omega t) = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = (M+m)2\dot{r}r\dot{\theta} + (M+m)r^2\ddot{\theta} + ma\dot{r}\Omega cos(\theta - \Omega t) - marsin(\theta - \Omega t)(\dot{\theta} - \Omega)$$

$$\rightarrow 0 = (M+m)2\dot{r}r\dot{\theta} + (M+m)r^2\ddot{\theta} + mar\Omega sin(\theta - \Omega t)$$

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For steady state set $\dot{r} = \ddot{r} = \ddot{\theta} = 0$. The two equations now give:

For r:

$$(M+m)r\dot{\theta}^2 - kr = -ma\Omega^2 cos(\theta - \Omega t)$$

$$\rightarrow r(k - (M+m)\dot{\theta}^2) = ma\Omega^2 cos(\theta - \Omega t)\ (1)$$

For θ:

$$mar\Omega sin(\theta - \Omega t) = 0 \; \forall t \rightarrow \theta = \Omega t + n\pi \; (2)$$

We see that if $\Omega^2 < k/(M+m) = \Omega_{res}^2$, then $\theta = \Omega t$, $r = \frac{ma\Omega^2}{k-(M+m)\Omega^2} > 0$ is a valid solution to both (1) and (2). But for $\Omega^2 > \Omega_{res}^2$, r becomes negative which does not fit our $r \geq 0$ constraint. So we need $\theta = \Omega t + \pi$, then $r = \frac{ma\Omega^2 cos(\pi)}{k-(M+m)\Omega^2} = \frac{ma\Omega^2}{(M+m)\Omega^2 - k} > 0$ solves both (1) and (2).