Prelims Solutions

Problem J12M1

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In equilibrium we have force balance, torque balance, and a constraint from the existence of the rigid rod. The constraint requires the vertical displacement of a post y_i to lie along the line of the rod: $y_i = px_i + b$ where p is the slope of the rod and b is the y-intercept (b is also the vertical displacement of the center of the rod). Torque balance gives

$$0 = \sum_{i=1}^{N} F_i x_i = \sum_{i=1}^{N} -ky_i x_i = -k\sum_{i=1}^{N}(px_i^2 + bx_i) \rightarrow p = -b\frac{\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i^2}$$

Force balance gives

$$Mg = \sum_{i=1}^{N} F_i = -k\sum_{i=1}^{N}(px_i + b) = -kp\sum_{i=1}^{N} x_i - kbN = kb[\frac{(\sum_{i=1}^{N} x_i)^2}{\sum_{i=1}^{N} x_i^2} - N] \rightarrow kb = \frac{Mg}{[\frac{(\sum_{i=1}^{N} x_i)^2}{\sum_{i=1}^{N} x_i^2} - N]}$$

Hence the force on the ith post is

$$F_i = -ky_i = -kpx_i - kb = kb[\frac{\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i^2}x_i - 1] \rightarrow$$

$$F_i = Mg\frac{\frac{\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i^2}x_i - 1}{\frac{(\sum_{i=1}^{N} x_i)^2}{\sum_{i=1}^{N} x_i^2} - N}$$

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Not in equilibrium we have $ma = F$, $\tau = I\ddot{\theta}$, where θ is the angle of the rod from the horizontal and $p = p(t)$, $b = b(t)$. So

$$M\ddot{b} = \sum_{i=1}^{N} F_i = -kp\sum_{i=1}^{N} x_i - kbN$$

For small oscillations $b \approx y_i$ and $y_i/x_i << 1$ so $p = (y_i - b)/x_i = tan(\theta) << 1 \rightarrow p \approx \theta$. Now torque gives

$$I\ddot{p} = -kp\sum_{i=1}^{N} x_i^2 - kb\sum_{i=1}^{N} x_i$$

This is now just 2 coupled harmonic oscillators. The question wants normal modes where b oscillates without p and p oscillates without b motion. We see that this occurs when the coupling term vanishes, which is the same for both equations:

$$\sum_{i=1}^{N} x_i = 0$$