M12T.3 Solution

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a) As with many thermodynamics problems we start by finding the partition function for a single microstate. Note that the probability of being at a certain relative θ is

$$P(\theta) = e^{-\beta S^2 J \cos\theta}$$

Thus the partition function is the sum of all these possible θ's, giving an integral since it's the continuous classical case. Note that we need to sum over both possible angles (azimuthal and polar). For simplicity I chose the xz-plane to be the plane containing the two vectors so that the angle between them is θ.

$$Z_1 = \int_0^{2\pi} \int_0^{\pi} e^{-\beta S^2 J \cos\theta} \sin\theta d\theta d\phi$$

$$Z_1 = 2\pi \int_1^{-1} e^{-\beta S^2 J u} du$$

$$Z_1 = \frac{2\pi}{\beta S^2 J} \left(e^{\beta S^2 J} - e^{-\beta S^2 J} \right)$$

$$Z_{total} = Z_1^N = \left(\frac{2\pi}{\beta S^2 J} \right)^N \left(e^{\beta S^2 J} - e^{-\beta S^2 J} \right)^N$$

Using the relation between the partition function and the energy we get

$$\langle H \rangle = -\frac{\partial}{\partial \beta} Z_{total}$$

We know the lefthand side from the given Hamiltonian

$$\langle H \rangle = -JN \langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle$$

Thus we can come up with an expression for $\langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle$. Using this I get

$$-JN \langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle = \frac{N}{\beta} - NS^2 J \coth \beta S^2 J$$

$$\langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle = -\frac{1}{J\beta} + S^2 \coth \beta S^2 J$$

b) The average specific heat per spin is given by

$$\begin{aligned} c(T) =& \frac{\frac{\partial}{\partial T}\langle H \rangle}{N} \\ c(T) =& -J\frac{\partial}{\partial T}\langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle \\ c(T) =& k_b - \frac{S^4 J^2}{k_b T^2}\,\text{csch}^2\left(\frac{S^2 J}{k_b T}\right) \end{aligned}$$

The estimate of using $\frac{\langle H \rangle}{N} = -J\langle \vec{S}_i \cdot \vec{S}_{i+1} \rangle$ becomes a good approximation as $N \to \infty$

c) to find the $T \to 0$ limit note that

$$\frac{S^4 J}{k_b T^2} csch^2\left(\frac{S^2 J}{k_b T}\right) \propto \frac{1}{T^2(e^{\frac{S^2 J}{k_b T}} - e^{\frac{-S^2 J}{k_b T}})^2}$$

Which goes to 0 as $T \to 0$ since the exponential increases much quicker than the polynomial.

$$\lim_{T \to 0} c(T) = k_b$$

To compare to the **quantum** ferromagnet I needed to derive it myself. TL;DR the quantum ferromagnet specific heat goes to zero so it is not consistent with the quantum version.

$$U \propto \int n^2 dn \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

Using $k = \frac{n\pi}{L}$ and $\omega \propto k^2$ we get

$$U \propto \int n^2 dn \frac{n^2}{e^{\beta n^2} - 1}$$

Letting $x = \beta n^2 \Rightarrow dn = \frac{dx}{2\beta n}$ we get

$$\begin{aligned} U \propto& \frac{1}{\beta}\int n^3 dx \frac{1}{e^{-x} - 1} = \frac{1}{\beta^{5/2}}\int dx \frac{x^{3/2}}{e^{-x} - 1} \\ U \propto& T^{5/2} \\ \Rightarrow C_Q(T) \propto& T^{3/2} \end{aligned}$$

Thus the correct quantum behavior is that c(T) does not go to a constant value as $T \to 0$ so the hamiltonian given is not consistent with the quantum ferromagnet