m12 #1.

a. Let E = thermal energy. This must be conserved:

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{q} \overset{!}{=} 0$$

where $\vec{q} = -K \vec{\nabla} T$ ← temperature

so $$\frac{\partial E}{\partial t} - K \nabla^2 T = 0$$

and $E = cT$ heat capacity, so writing temperature as $U(t,z)$,

$$\frac{\partial U}{\partial t} = \frac{K}{c} \nabla^2 U = \alpha^2 \nabla^2 U \quad \text{heat equation!}$$

solve with separation of variables: $U(t,z) = T(t) Z(z)$

$$T' Z = \alpha^2 T Z'' \longrightarrow \frac{T'}{T} = \alpha^2 \frac{Z''}{Z} = \text{constant.}$$

suppose the constant is complex but nonzero ($c = 0$ would not fit b.c.)

$$\rightarrow \quad T(t) = A e^{ct}$$

$$Z(z) = B e^{\sqrt{\frac{c}{\alpha^2}} z} + C e^{-\sqrt{\frac{c}{\alpha^2}} z}$$

so $U(t,z) = e^{ct}\left(A e^{\sqrt{\frac{c}{\alpha^2}} z} + B e^{-\sqrt{\frac{c}{\alpha^2}} z}\right) + T_0 \quad A, B, c \in \mathbb{C}$

boundary condition: $U(t,0) = T_0 - T_1 \cos\omega t$

$$e^{ct}(A+B) = -T_1 \cos\omega t$$

Let $c = i\omega$, then if $(A+B) = -T_1$, $\mathrm{Re}[U(t,z)] = \mathrm{Re}\left[T_0 - T_1 e^{i\omega t}\right]$

Now $U(t,z) = \mathrm{Re}\left[e^{i\omega t}\left(A e^{\sqrt{i}\sqrt{\frac{\omega}{\alpha^2}} z} + B e^{-\sqrt{i}\sqrt{\frac{\omega}{\alpha^2}}}\right)\right]$ with $A + B = -T_1$

note $\sqrt{i} = \pm \frac{1}{\sqrt{2}}(1+i)$, so

$$U(t,z) = \mathrm{Re}\left[e^{i\omega t}\left(A e^{\pm\sqrt{\frac{\omega}{2\alpha^2}} z} e^{\pm i\sqrt{\frac{\omega}{2\alpha^2}} z} + B e^{\mp\sqrt{\frac{\omega}{2\alpha^2}} z} e^{\mp i\sqrt{\frac{\omega}{2\alpha^2}} z}\right)\right]$$

let $a = \sqrt{\frac{\omega}{2\alpha^2}}$

$$= \mathrm{Re}\left[(\cos\omega t + i\sin\omega t)\left(A e^{-az}(\cos az - i\sin az) + B e^{-az}(\cos az - i\sin az)\right)\right]$$

where above we ignored unphysical terms like $\sim e^{az}$.

Also, we can let $B = 0$ without loss of generality, so

$$U(t,z) = T_0 - T_1 e^{-az}(\cos\omega t \cos az + \sin\omega t \sin az)$$

I plugged this back into $\frac{\partial U}{\partial t} = \alpha^2 \frac{\partial^2 U}{\partial z^2}$ and it works and satisfies the B.C.

See next page for correction. Though uncrossed out portion on this page is correct.

b. Recall $\dfrac{\partial E}{\partial t} + \nabla \cdot \vec{q} = 0$

No heat transfer at surface $\longleftrightarrow$ $\nabla \cdot q = 0 \rightarrow \dfrac{\partial E}{\partial t} = 0 \rightarrow \dfrac{\partial u(t,0)}{\partial t} = 0$

$u(t,0) = T_0 - T_1 \cos \omega t$

$\dfrac{\partial u}{\partial t} = T_1 \sin \omega t = 0 \rightarrow \omega t = 0, \pi.$

We are told the min surface temp occurs at 4AM and max temp at 4PM. By inspection of $u(t,0)$, this tells us that $\omega t = 0 \longrightarrow$ 4 AM

and $\omega t = \pi \longrightarrow$ 4 PM.

~~So 4AM, 4PM are also the times when $\nabla \cdot q = 0$.~~

c. Recall our solution

$$u(t,z) = T_0 z - T_1 e^{-az} (\cos \omega t \cos az + \sin \omega t \sin kz)$$

with $a = \sqrt{\dfrac{\omega}{2\alpha^2}} = \sqrt{\dfrac{\omega c}{2K}}$.

a measures how far the temperature penetrates into the ground.

~~Since a has no dependence on T_1, I believe the ratio between penetration due to daily cycle : penetration due to yearly cycle should equal 1.~~

Note a depends on ω. For the daily cycle, $\omega \sim \dfrac{1}{24 \text{ hrs}} = \dfrac{1}{1 \text{ day}}$

For a yearly cycle $\omega \sim \dfrac{1}{365 \text{ days}}$.

Hence $\dfrac{a_{day}}{a_{year}} \sim \sqrt{\dfrac{\omega_{day}}{\omega_{year}}} \sim \sqrt{365}$

I don't know why they gave us the information about T_1 being $\sim$ same magnitude in both cycles.

Final note: can rewrite $u(t,z) = T_0 - T_1 e^{-az} \cos(\omega t + az)$ which might make it easier to verify it satisfies the equation.

b. $\vec{q} = -K\vec{\nabla}T = 0 \rightarrow \vec{\nabla}T = 0$

solve $\left.\frac{\partial v}{\partial z}\right|_{z=0} = 0.$

$$v(t,z) = T_0 - T_1 e^{-az}\cos(\omega t + az)$$

$$\frac{\partial v}{\partial z} = -T_1\left[-ae^{-az}\cos(\omega t + az) - e^{az}\sin(\omega t + az)a\right] = 0$$

$$\cos(\omega t) + \sin(\omega t) = 0$$

at $\omega t = \frac{3\pi}{4}, \frac{7\pi}{4}$

$\longleftrightarrow \frac{3}{4}(12 \text{ hour}) + 4 \text{ AM} \rightarrow$ 1 PM , 1 AM