May 2012 Quantum Problem 3
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3. A system of two massive particles, of spins \( s_a > 0 \) and \( s_b > 0 \), is governed by the Hamiltonian:

\[
H = K + V(|\vec{r}_a - \vec{r}_b|) + f(|\vec{r}_a - \vec{r}_b|) \left(S_z^{(a)} - S_z^{(b)}\right),
\]

with \( K \) the usual kinetic energy operator and \( S_z^{(a)} \) and \( S_z^{(b)} \) the spin operators. The function \( f \) is negative at all distances: \( f(r) < 0 \), and the interaction potential \( V(r) \) is finite and sufficiently attractive so that the system has at least one bound state. Let \( \vec{S} = \vec{S}^{(a)} + \vec{S}^{(b)} \) be the total spin operator.

a) Explain why this system’s ground state is non-degenerate.

b) What are the ground-state expectation values of the total spin’s component \( S_z \), and of the total spin operator \( |\vec{S}|^2 \)? For which of these operators is the ground state also an eigenfunction?

c) Consider now the case \( s_a = 1, s_b = 1/2 \). List the possible values of \( |\vec{S}|^2 \) and \( S_z \). What are the probabilities of observing these outcomes when \( |\vec{S}|^2 \) and \( S_z \) are measured in the system’s ground state?

\[\]

\textbf{Part a)}

The ground state is determined when \( H \) is at its lowest possible energy. Our energy levels are set by the eigenvalues of \( S_z^{(a)} \) and \( S_z^{(b)} \), namely: \( m_a \in \{-s_a, -s_a + 1, ..., s_a - 1, s_a\} \) and \( m_b \in \{-s_b, -s_b + 1, ..., s_b - 1, s_b\} \). In our case \( f(r) < 0 \), so a minimum energy is \textit{uniquely} achieved when \( m_a = s_a \) and \( m_b = -s_b \), making our ground state non-degenerate.

\textbf{Part b)}

Call the ground state \( |0\rangle \). From part a, we see that the ground state is an eigenfunction of \( S_z \) and have:

\[
\langle S_z \rangle = \langle 0 | (S_a + S_b) | 0 \rangle = \langle 0 | S_a | 0 \rangle + \langle 0 | S_b | 0 \rangle = \hbar (s_a - s_b) \langle 0 | 0 \rangle = \hbar (s_a - s_b)
\]

In a tragic turn of events, however, the ground state is not an eigenfunction of \( |\vec{S}|^2 \), and we will need to invoke everyone’s two favorite uncles, Clebsch and Gordan. Per Griffiths eq. 4.184:

\[
|s_a s_b m_a m_b\rangle = \sum_s C_{m_a m_b s}^{s_a s_b} |s m\rangle, \quad (m = m_a + m_b)
\]

In our case, the ground state is:

\[
|s_a s_b s_a (-s_b)\rangle = \sum_s C_{s_a (-s_b) s}^{s_a s_b} (s_a - s_b) |s (s_a - s_b)\rangle
\]

We therefore have the sad reality that:
\( \langle S^2 \rangle = \langle s_a s_b s_a (-s_b) | S^2 | s_a s_b s_a (-s_b) \rangle = \hbar^2 \sum_s s(s + 1) \left[ C^{s_a s_b s}_{s a (-s_b)(s_a - s_b)} \right]^2 \)

**Part c)**

Before considering we are in the ground state, possible values for \( S_z \) are \(-\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \text{and} \frac{3\hbar}{2}\)

and values for \( S^2 \) are \( \frac{3\hbar^2}{4} \) and \( \frac{15\hbar^2}{4} \) (depending on if total spin were \( \frac{3}{2} \) or \( \frac{1}{2} \)).

Now, because the ground state is an eigenfunction of \( S_z \) (with eigenvalues determined in part a), we will obtain \( S_z = \hbar(s_a - s_b) = \frac{\hbar}{2} \) with 100% probability (so all other possibilities have probability 0).

In the case of \( S^2 \), we write down the clebsch-gordon coefficients that were clearly not an extensive waste of time to memorize:

\[ s_a = 1 \text{ and } s_b = 1/2 \Rightarrow |0\rangle = |1 \frac{1}{2} 1 \frac{-1}{2} \rangle = \sqrt{\frac{1}{3}} |1 \frac{3}{2} 1 \frac{1}{2} \rangle + \sqrt{\frac{2}{3}} |1 \frac{1}{2} 1 \frac{1}{2} \rangle \]

Therefore implying that we could obtain

- \( S^2 = \hbar^2 \left( \frac{3}{2} + 1 \right) \left( \frac{3}{2} \right) = \frac{15\hbar^2}{4} \) with a probability of \( \frac{1}{3} \)
- \( S^2 = \hbar^2 \left( \frac{1}{2} + 1 \right) \left( \frac{1}{2} \right) = \frac{3\hbar^2}{4} \) with a probability of \( \frac{2}{3} \)