May 2012 Quantum Problem 3

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3. A system of two massive particles, of spins $s_a > 0$ and $s_b > 0$, is governed by the Hamiltonian:

$$H = K + V(|\vec{r}_a - \vec{r}_b|) + f(|\vec{r}_a - \vec{r}_b|)\left(S_z^{(a)} - S_z^{(b)}\right) ,$$

with K the usual kinetic energy operator and $\vec{S}^{(a)}$ and $\vec{S}^{(b)}$ the spin operators. The function f is negative at all distances: $f(r) < 0$, and the interaction potential $V(r)$ is finite and sufficiently attractive so that the system has at least one bound state. Let $\vec{S} = \vec{S}^{(a)} + \vec{S}^{(b)}$ be the total spin operator.

a) Explain why this system's ground state is non-degenerate.

b) What are the ground-state *expectation values* of the total spin's component S_z, and of the total spin operator $|\vec{S}|^2$? For which of these operators is the ground state also an eigenfunction?

c) Consider now the case $s_a = 1$, $s_b = 1/2$. List the possible values of $|\vec{S}|^2$ and S_z. What are the probabilities of observing these outcomes when $|\vec{S}|^2$ and S_z are measured in the system's ground state?

Part a)

The ground state is determined when H is at its lowest possible energy. Our energy levels are set by the eigenvalues of $S_z^{(a)}$ and $S_z^{(b)}$, namely: $m_a \in \{-s_a, -s_a + 1, ..., s_a - 1, s_a\}$ and $m_b \in \{-s_b, -s_b + 1, ..., s_b - 1, s_b\}$. In our case $f(r) < 0$, so a minimum energy is *uniquely* achieved when $m_a = s_a$ and $m_b = -s_b$, making our ground state non-degenerate

Part b)

Call the ground state $|0\rangle$. From part a, we see that $\boxed{\text{the ground state is an eigenfunction of } S_z}$ and have:

$$\langle S_z \rangle = \langle 0| (S_a + S_b) |0\rangle = \langle 0| S_a |0\rangle + \langle 0| S_b |0\rangle = \hbar(s_a - s_b) \langle 0||0\rangle = \boxed{\hbar(s_a - s_b)}$$

In a tragic turn of events, however, $\boxed{\text{the ground state is not an eigenfunction of } |S|^2}$, and we will need to invoke everyone's two favorite uncles, Clebsch and Gordan. Per Griffiths eq. 4.184:

$$|s_a\, s_b\, m_a\, m_b\rangle = \sum_s C^{s_a\, s_b\, s}_{m_a\, m_b\, m} |s\, m\rangle, \ \ (m = m_a + m_b)$$

In our case, the ground state is:

$$|s_a\, s_b\, s_a\, (-s_b)\rangle = \sum_s C^{s_a\, s_b\, s}_{s_a\, (-s_b)\, (s_a - s_b)} |s\, (s_a - s_b)\rangle$$

We therefore have the sad reality that:

$$\langle S^2\rangle = \langle s_a\, s_b\, s_a\, (-s_b)|\, S^2\, |s_a\, s_b\, s_a\, (-s_b)\rangle = \boxed{\hbar^2 \sum_s s(s+1)\left[C^{s_a\, s_b\, s}_{s_a\, (-s_b)\, (s_a-s_b)}\right]^2}$$

Part c)

Before considering we are in the ground state, possible values for S_z are $\boxed{\frac{-3\hbar}{2}, \frac{-\hbar}{2}, \frac{\hbar}{2}, \text{ and } \frac{3\hbar}{2}}$ and values for S^2 are $\boxed{\frac{3\hbar^2}{4} \text{ and } \frac{15\hbar^2}{4}}$ (depending on if total spin were $\frac{3}{2}$ or $\frac{1}{2}$).

Now, because the ground state is an eigenfunction of S_z (with eigenvalues determined in part a), we will obtain $\boxed{S_z = \hbar(s_a - s_b) = \frac{\hbar}{2} \text{ with 100\% probability}}$ (so all other possibilities have probability 0).

In the case of S^2, we write down the clebsch-gordon coefficients that were clearly not an extensive waste of time to memorize:

$$s_a = 1 \text{ and } s_b = 1/2 \Rightarrow |0\rangle = |1\ \frac{1}{2}\ 1\ \frac{-1}{2}\rangle = \sqrt{\frac{1}{3}}|\frac{3}{2}\ \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|\frac{1}{2}\ \frac{1}{2}\rangle$$

Therefore implying that we could obtain

- $S^2 = \hbar^2\left(\frac{3}{2}+1\right)\left(\frac{3}{2}\right) = \boxed{\frac{15\hbar^2}{4}}$ with a probability of $\boxed{\frac{1}{3}}$
- $S^2 = \hbar^2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}\right) = \boxed{\frac{3\hbar^2}{4}}$ with a probability of $\boxed{\frac{2}{3}}$