M12Q.2 - Tritium Decay (Solution by Jim Wu)

Tritium (^{3}H, a radioactive isotop of hydrogen) decays to ^{3}He with the emission of an electron (and an antineutrino). Assume that this decay and electron emission is rapid enough so that as far as the other electron is concerned all that happens is that the atom's nucleus instantaneously changes its charge from $+e$ to $+2e$.

(a) Write the normalized ground state wave function for the one-electron atom or ion with nuclear charge Ze, neglecting spin and other fine-structure or relativistic effects. You may take it as given that the wave function is of the form $\psi(r,\theta,\phi) = Ae^{-\kappa r}$ with A and κ dependent on the relevant parameters.

(b) Assuming the tritium atom was originally in its ground state, what is the probability of finding, immediately after the decay, the resulting He^+ ion in its ground state?

(c) In the event that the resulting He^+ ion is not its ground state, compute its average excitation energy relative to the ion's ground state.

Solution:

(a) The normalized ground state wave function for a hydrogenic atom with nuclear charge Ze is

$$\psi(r,\theta,\phi) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\sqrt{\frac{1}{4\pi}}$$

where the Bohr radius is $a_0 = \frac{\hbar}{m_e c\alpha}$ and $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$ is the fine structure constant.

(b) The ground state of the tritium with $Z = 1$ is

$$\psi_T = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\sqrt{\frac{1}{4\pi}}$$

and the ground state for the He+ ion with $Z = 2$ is

$$\psi_{He} = 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}\sqrt{\frac{1}{4\pi}}$$

The probability amplitude of finding the helium ion in its ground state after the decay is

$$\begin{aligned}
\langle\psi_{He}|\psi_T\rangle &= 4\left(\frac{2}{a_0^2}\right)^{3/2}\int_0^\infty r^2 e^{-3r/a_0}\,dr \\
&= 4\left(\frac{2}{a_0^2}\right)^{3/2}\left(\frac{a_0}{3}\right)^3\int_0^\infty y^2 e^{-y}\,dy \\
&= 4\left(\frac{2}{a_0^2}\right)^{3/2}\left(\frac{a_0}{3}\right)^3(2!) \\
&= \frac{2^{9/2}}{3^3}
\end{aligned}$$

Hence, the probability of finding the helium ion in the ground state is

$$P = |\langle \psi_{He} | \psi_T \rangle|^2 = \boxed{\frac{2^9}{3^6}} \approx 0.702 \,.$$

(c) The average excitation energy relative to the ion's ground state is

$$\tilde{E} = \left(\sum_{n>1}^{\infty} |c_n|^2 E_n \right) - E_1 = \left(\sum_{n=1} |c_n|^2 E_n \right) - (1 + c_1) E_1 = \langle E \rangle - (1 + c_1) E_1$$

where E_1 is the ground state energy of the helium ion.

Since the tritium was originally in the *s* orbital, then after the decay, we must have the helium ion be in the *s* orbital as well. There is zero probability to be in any other angular momentum state. So, the only contribution to the average energy comes from the radial part of the wave function.

For the helium ion, the radial part of the Hamiltonian is given by

$$H_R = -\frac{\hbar^2}{2m} \nabla^2 - \frac{2e^2}{4\pi\epsilon_0 r}$$

The average energy of the electron originally in the ground state of the tritium atom is

$$\begin{aligned}
\langle E \rangle &= 4\pi \int_0^\infty \psi_T^* H_r \psi_T \, r^2 dr \\
&= 4 \left(\frac{1}{a_0} \right)^3 \int_0^\infty dr \, r^2 e^{-r/a_0} \left(-\frac{\hbar^2}{2mr^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{2e^2}{4\pi\epsilon_0 r} \right) e^{-r/a_0} \\
&= 4 \left(\frac{1}{a_0} \right)^3 \int_0^\infty \left(\frac{\hbar^2 r}{m a_0} - \frac{\hbar^2 r^2}{2 m a_0^2} - \frac{2e^2}{4\pi\epsilon_0} r \right) e^{-2r/a_0} \, dr \\
&= 4 \left(\frac{1}{a_0} \right)^3 \left[\frac{\hbar^2}{m a_0} \left(\frac{a_0}{2} \right)^2 \int_0^\infty y e^{-y} \, dy - \frac{\hbar^2}{2 m a_0^2} \left(\frac{a_0}{2} \right)^3 \int_0^\infty y^2 e^{-y} \, dy - \frac{2e^2}{4\pi\epsilon_0} \left(\frac{a_0}{2} \right)^2 \int_0^\infty y e^{-y} \, dy \right] \\
&= 4 \left(\frac{1}{a_0} \right)^3 \left(\frac{\hbar^2 a_0}{4m} - \frac{\hbar^2 a_0}{8m} - \frac{e^2 a_0^2}{8\pi\epsilon_0} \right) \\
&= \frac{\hbar^2}{2 m a_0^2} - \frac{e^2}{2\pi\epsilon_0 a_0} \\
&= \frac{m_e^2 c^2 \alpha^2}{2} - 2 m_e c^2 \alpha^2 \\
&= -\frac{3}{2} m_e c^2 \alpha^2
\end{aligned}$$

Recall that the energy of the ground state of a hydrogen atom is $E_1 = -\frac{1}{2} m_e c^2 \alpha^2 = -13.6$ eV. So, the average energy of the helium ion is

$$\langle E \rangle = 3(-13.6 \text{ eV})$$

So the average excitation energy relative to the ground state energy of the helium ion is

$$\begin{aligned}\tilde{E} &= 3(-13.6\text{ eV}) - \left(1 + \frac{2^9}{3^6}\right) 4(-13.6\text{ eV}) \\ &= -(-13.6\text{ eV})\left(1 + \frac{2^{11}}{3^6}\right) \\ &\approx 51.8\text{ eV}\end{aligned}$$

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