Princeton Physics Prelims: M12Q.1 Scattering from finite radial potential

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See also: Townsend's "A Modern Approach to Quantum Mechanics" example on pg. 471 which is identical/nearly the same, and J03Q.1 which is similar.

1 Find $d\sigma/d\Omega$ and σ in the limit of zero incident energy

"Low incident energy" means we want to use a partial wave expansion, and only consider s-wave scattering (angular momentum $\ell = 0$) since it is the only wave component in the spherical wave expansion that will have appreciable scattering. With the standard substitution $u = r \cdot R(r)$, the radial Schrodinger equation inside and outside the potential is:

$$(r < a) \qquad -\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} - V_0 u = Eu \tag{1}$$

$$(r > a) \qquad -\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} = Eu \tag{2}$$

To find general solutions, rewrite the above equations as

$$(r < a) \qquad \frac{d^2u}{dr^2} = -\frac{2\mu}{\hbar^2}(E + V_0)u \equiv -k_0^2 u, \qquad k_0 \equiv \sqrt{\frac{2\mu}{\hbar^2}(E + V_0)} \tag{3}$$

$$(r > a) \qquad \frac{d^2u}{dr^2} \equiv -k^2 u, \qquad k \equiv \sqrt{\frac{2\mu}{\hbar^2}E} \tag{4}$$

Furthermore, $u(r = 0) = 0 \cdot R(r) = 0$, prompting us to write $u(r) = A\sin(k_0 r)$ for $r < a$ (A is a constant). In partial wave expansion we treat s-wave scattering as equivalent to a phase shift, so $u(r) = C\sin(kr + \delta_0)$ for $r > a$ (C is a constant).

Continuity of $u(r)$ and $u'(r)$ at $r = a$ give, respectively,

$$A\sin(k_0 a) = C\sin(ka + \delta_0) \tag{5}$$

$$Ak_0\cos(k_0 a) = Ck\cos(ka + \delta_0) \tag{6}$$

Taking the ratio of these equations,

$$\frac{1}{k_0}\tan(k_0 a) = \frac{1}{k}\tan(ka + \delta_0) \tag{7}$$

In the low incident energy limit, $\tan(ka + \delta_0) \approx ka + \delta_0$, so the previous equation becomes

$$\frac{1}{k_0}\tan(k_0 a) = \frac{1}{k}(ka + \delta_0) \tag{8}$$

$$\delta_0 = \frac{k}{k_0}\tan(k_0 a) - ka \tag{9}$$

$$\delta_0 = ka\left(\frac{\tan(k_0 a)}{k_0 a} - 1\right) \tag{10}$$

The partial wave expansion states that if we know the phase shifts δ_ℓ, the scattering amplitude $f(\theta)$ can be expanded as

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\frac{e^{i\delta_\ell}}{k}\sin(\delta_\ell)P_\ell(\cos\theta) \tag{11}$$

For prelims we just have to remember that in s-wave scattering ($\ell = 0$), $f(\theta) = \frac{e^{i\delta_0}}{k}\sin(\delta_0)$. The differential cross-section is

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 = \frac{1}{k^2}\sin^2\left[ka\left(\frac{\tan(k_0 a)}{k_0 a}\right) - 1\right] \tag{12}$$

Again using the low-energy (small ka) approximation,

$$\frac{d\sigma}{d\Omega} \approx \frac{1}{k^2}k^2 a^2\left(\frac{\tan(k_0 a)}{k_0 a} - 1\right)^2 \tag{13}$$

$$\boxed{\frac{d\sigma}{d\Omega} \approx a^2\left(\frac{\tan(k_0 a)}{k_0 a} - 1\right)^2} \tag{14}$$

Integrating over solid angle to get the total cross-section:

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega \tag{15}$$

$$\boxed{\sigma = 4\pi a^2\left(\frac{\tan(k_0 a)}{k_0 a} - 1\right)^2} \tag{16}$$

(b) What values of V_0 causes this cross-section to diverge to ∞? What is the physical significance of such divergences of σ?

The tangent term diverges when $k_0 a = (2l+1)\pi/2$, $l = 0, 1, 2, \ldots$. Substituting this back into our definition of k_0 in Eqn. (3), we get

$$V_{0,divergence} = \frac{\hbar^2}{2\mu}k_{0,l}^2 - E = \frac{\hbar^2}{2\mu}\left((2l+1)\frac{\pi}{2a}\right)^2 - E, \qquad (l = 0, 1, 2, 3, \ldots) \tag{17}$$

Infinite cross-section means that the particles will always interact/scatter with the potential. This leads me to think that there is some kind of resonance for these V_0, given an incident energy E.