Princeton Physics Prelims: M12Q.1 Scattering from finite radial potential

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See also: Townsend’s “A Modern Approach to Quantum Mechanics” example on pg. 471 which is identical/nearly the same, and J03Q.1 which is similar.

1 Find \( d\sigma/d\Omega \) and \( \sigma \) in the limit of zero incident energy

“Low incident energy” means we want to use a partial wave expansion, and only consider \( s \)-wave scattering (angular momentum \( \ell = 0 \)) since it is the only wave component in the spherical wave expansion that will have appreciable scattering. With the standard substitution \( u = r \cdot R(r) \), the radial Schrödinger equation inside and outside the potential is:

\[
(r < a) \quad -\frac{\hbar^2}{2\mu} \frac{d^2 u}{dr^2} - V_0 u = Eu \\
(r > a) \quad -\frac{\hbar^2}{2\mu} \frac{d^2 u}{dr^2} = Eu
\]

To find general solutions, rewrite the above equations as

\[
(r < a) \quad \frac{d^2 u}{dr^2} = -\frac{2\mu}{\hbar^2} (E + V_0)u \equiv -k_0^2 u, \quad k_0 \equiv \sqrt{\frac{2\mu}{\hbar^2} (E + V_0)}
\]

\[
(r > a) \quad \frac{d^2 u}{dr^2} = -k^2 u, \quad k \equiv \sqrt{\frac{2\mu}{\hbar^2} E}
\]

Furthermore, \( u(r = 0) = 0 \cdot R(r) = 0 \), prompting us to write \( u(r) = A \sin(k_0 r) \) for \( r < a \) (\( A \) is a constant). In partial wave expansion we treat \( s \)-wave scattering as equivalent to a phase shift, so \( u(r) = C \sin(kr + \delta_0) \) for \( r > a \) (\( C \) is a constant).

Continuity of \( u(r) \) and \( u'(r) \) at \( r = a \) give, respectively,

\[
A \sin(k_0 a) = C \sin(ka + \delta_0) \tag{5}
\]

\[
Ak_0 \cos(k_0 a) = Ck \cos(ka + \delta_0) \tag{6}
\]

Taking the ratio of these equations,

\[
\frac{1}{k_0} \tan(k_0 a) = \frac{1}{k} \tan(ka + \delta_0) \tag{7}
\]
In the low incident energy limit, \( \tan(ka + \delta_0) \approx ka + \delta_0 \), so the previous equation becomes

\[
\frac{1}{k_0} \tan(k_0a) = \frac{1}{k}(ka + \delta_0)
\]

(8)

\[
\delta_0 = \frac{k}{k_0} \tan(k_0a) - ka
\]

(9)

\[
\delta_0 = ka \left( \frac{\tan(k_0a)}{k_0a} - 1 \right)
\]

(10)

The partial wave expansion states that if we know the phase shifts \( \delta_\ell \), the scattering amplitude \( f(\theta) \) can be expanded as

\[
f(\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) \frac{e^{i\delta_\ell}}{k} \sin(\delta_\ell) P_\ell(\cos \theta)
\]

(11)

For prelims we just have to remember that in \( s \)-wave scattering (\( \ell = 0 \)), \( f(\theta) = \frac{e^{i\delta_0}}{k} \sin(\delta_0) \). The differential cross-section is

\[
d\sigma = |f(\theta)|^2 = \frac{1}{k^2} \sin^2 \left[ ka \left( \frac{\tan(k_0a)}{k_0a} \right) - 1 \right]
\]

(12)

Again using the low-energy (small \( ka \)) approximation,

\[
\frac{d\sigma}{d\Omega} \approx \frac{1}{k^2} a^2 \left( \frac{\tan(k_0a)}{k_0a} - 1 \right)^2
\]

(13)

\[
\frac{d\sigma}{d\Omega} \approx a^2 \left( \frac{\tan(k_0a)}{k_0a} - 1 \right)^2
\]

(14)

Integrating over solid angle to get the total cross-section:

\[
\sigma = \int \frac{d\sigma}{d\Omega} d\Omega
\]

(15)

\[
\sigma = 4\pi a^2 \left( \frac{\tan(k_0a)}{k_0a} - 1 \right)^2
\]

(16)

(b) What values of \( V_0 \) causes this cross-section to diverge to \( \infty \)?

What is the physical significance of such divergences of \( \sigma \)?

The tangent term diverges when \( k_0a = (2\ell + 1)\pi/2, \ell = 0, 1, 2, \ldots \). Substituting this back into our definition of \( k_0 \) in Eqn. (3), we get

\[
V_{0,\text{divergence}} = \frac{\hbar^2}{2\mu} k_{0,l}^2 - E = \frac{\hbar^2}{2\mu} \left( (2\ell + 1) \frac{\pi}{2a} \right)^2 - E, \quad (\ell = 0, 1, 2, 3, \ldots)
\]

(17)

Infinite cross-section means that the particles will always interact/scatter with the potential. This leads me to think that there is some kind of resonance for these \( V_0 \), given an incident energy \( E \).