M12M.3 Solution

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I decided to go with Newtonian methods to find the equation of motion in θ. Let the x and y axes be the horizontal and vertical axes of the diagram pictured in the problem. At equilibrium we get the following force equations

$$\sum F_y = 0 : mg = F_N \cos\theta_0 + F_f \sin\theta_0 \tag{1}$$
$$\sum F_x = 0 : F_f \cos\theta_0 = F_N \sin\theta_0 \tag{2}$$

and of course we get the fundamental friction relation

$$F_f = \mu F_N \tag{3}$$

equations (2) and (3) immediately give us our equation for θ_0

$$\mu = \tan\theta_0 \tag{4}$$

To find the frequency of small oscillations I need the equation of motion for general θ

$$I\ddot{\theta} = \tau_g + \tau_f \tag{5}$$
$$ma^2\ddot{\theta} = -mga\sin\theta + aF_f \tag{6}$$

Now that we have to think about non-equilibrium motion, equations (1) and (2) don't apply here. To get a formula for F_f I found it easier to rotate my coordinate system so the y'-axis lies anti-parallel to the radial vector and the x'-axis is along the azimuthal vector. Then I use the fact that the particle never moves in the y' direction (even when not in equilibrium). Thus I get the relation

$$F_g \cos\theta = F_N = \frac{F_f}{\mu} \tag{7}$$

Putting this back into equation (6) I get

$$ma^2\ddot{\theta} = -mga\sin\theta + amg\mu\cos\theta \tag{8}$$

Now I make the perturbation $\theta \to \theta_0 + \epsilon$

$$\ddot{\epsilon} = \frac{g}{a}(\mu \cos(\theta_0 + \epsilon) - \sin(\theta_0 + \epsilon)) \tag{9}$$

Using angle addition formulas

$$\begin{aligned}
\cos(\theta_0 + \epsilon) &= \cos(\theta_0)\cos(\epsilon) - \sin(\theta_0)\sin(\epsilon) &(10)\\
\cos(\theta_0 + \epsilon) &\approx \cos(\theta_0) - \sin(\theta_0)\epsilon &(11)\\
\sin(\theta_0 + \epsilon) &= \cos(\theta_0)\sin(\epsilon) + \sin(\theta_0)\cos(\epsilon) &(12)\\
\sin(\theta_0 + \epsilon) &\approx \cos(\theta_0)\epsilon + \sin(\theta_0) &(13)
\end{aligned}$$

Plugging these relations back into equation (9)

$$\ddot{\epsilon} = \frac{g}{a}(\mu(\cos(\theta_0) - \sin(\theta_0)\epsilon) - \cos(\theta_0)\epsilon - \sin(\theta_0)) \tag{14}$$

then, using $\mu = \tan\theta_0$, we get the final differential equation for ϵ

$$\begin{aligned}
\ddot{\epsilon} &= -\frac{g}{a\cos\theta_0}\epsilon &(15)\\
\Rightarrow \omega &= \sqrt{\frac{g}{a\cos\theta_0}} &(16)
\end{aligned}$$

To make the answer in term of the given quantities

$$\begin{aligned}
\cos\left(\tan^{-1}\mu\right) &= \frac{1}{\sqrt{1+\mu^2}} &(17)\\
\Rightarrow \omega &= \sqrt{\frac{g}{a}}(1+\mu^2)^{1/4} &(18)
\end{aligned}$$