Princeton Physics Prelims: M12E.1/J02E.2 Rotating Dielectric Cylinder Around Line of Charge in External Magnetic Field

Stephanie Kwan (skwan@princeton.edu)

January 25, 2019

(a) Compute the torque on the cylinder in terms of dB_z/dt. $B_z(t)$ is the (approx. uniform) axial magnetic field within the cylinder.

The changing magnetic field causes an electric field which produces a force (and thus a torque) on the charged cylinder. Faraday's Law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Curl in cylindrical coordinates is

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{\rho}/\rho & \hat{\phi} & \hat{z}/\rho \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix} \tag{1}$$

$\vec{E}$ is only in the $\hat{\phi}$ direction by (approx.) translational symmetry along the $\hat{z}$ axis. Denoting the time derivative of B_z as $\dot{B}_z$ and reading off the $\hat{z}$-component of the above determinant,

$$\frac{\partial}{\partial \rho}(\rho E_\phi) \cdot \frac{\hat{z}}{\rho} = -\dot{B}_z \tag{2}$$

$$\frac{\partial}{\partial \rho}(\rho E_\phi) = -\dot{B}_z \rho \hat{z} \tag{3}$$

$$\rho E_\phi = -\dot{B}_z \rho^2/2 \tag{4}$$

$$E_\phi = -\dot{B}_z \rho/2 \tag{5}$$

Thus the torque on the cylinder is

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times q\vec{E} = \vec{\rho} \times (+\lambda h) \cdot \left(-\frac{\dot{B}_z \rho}{2} \hat{\phi} \right) \tag{6}$$

$$\boxed{\vec{\tau} = -\frac{\lambda h \dot{B}_z r^2}{2} \hat{z}} \tag{7}$$

(b) Find the angular velocity of the cylinder after the external $\vec{B}$ is reduced to 0, noting that the final field in the cylinder is non-zero.

Torque is $\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$. $\omega_i = 0$, and we want to find ω_f. The torque per unit length is

$$-\frac{\lambda h \dot{B}_z r^2}{2} = I\dot{\omega} \tag{8}$$

$$\int d\omega = -\frac{\lambda r^2}{2I}\int dB_z \tag{9}$$

$$\omega_f = -\frac{\lambda r^2}{2I}(B_f - B_i) \tag{10}$$

$B_i = B_0\hat{z}$ because neither the line of charge nor the cylinder are rotating. To find B_f we use Ampere's law,

$$\vec{B}_f = \frac{\mu_0}{4\pi}I\int \frac{d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^{\,2}} = \frac{\mu_0}{4\pi}(\lambda\omega_f r)\int_0^{2\pi}\frac{r d\theta\hat{\theta}\times(+\vec{r})}{r^2} = \frac{\mu_0}{4\pi}\lambda\omega_f\cdot 2\pi\hat{z} = \frac{\mu_0}{2}\lambda\omega_f\hat{z} \tag{11}$$

$$\omega_f = -\frac{\lambda r^2}{2I}\left(\frac{\mu_0}{2}\lambda\omega_f - B_0\right) \tag{12}$$

$$\boxed{\omega_f = \frac{2\lambda r^2 B_0}{\mu_0\lambda^2 r^2 + 4I}} \tag{13}$$

(c) Express the angular momentum of the initial state. Demonstrate that the total angular momentum (mechanical + EM) is conserved between initial and final states.

The Poynting vector is $\vec{S} = 1/\mu_0(\vec{E}\times\vec{B})$. The linear momentum density is $\vec{g} = \mu_0\epsilon_0\vec{S} = \epsilon_0(\vec{E}\times\vec{B})$. Thus the angular momentum density is $\vec{\ell} = \vec{r}\times\vec{g}$, and we want to integrate this over an area to get angular momentum per unit length, which we denote as $\vec{L}$.

Use Gauss's Law for a cylinder surrounding the line of charge to find the initial E field inside the cylinder:

$$\int \vec{E}\cdot d\vec{a} = \rho_{encl}/\epsilon_0 \tag{14}$$

$$E_i\cdot 2\pi r\, d\ell = -\lambda\cdot d\ell/\epsilon_0 \tag{15}$$

$$\vec{E}_i = -\frac{\lambda}{2\pi r\epsilon_0}\hat{r}, \quad \vec{B}_i = B_0\hat{z} \tag{16}$$

Integrating the angular momentum density over a cross-sectional area of the cylinder:

$$\vec{L}_i = \int \vec{r}\times\vec{g}\, dA = \iint \vec{r}\times\epsilon_0\left(\frac{\lambda}{2\pi r\epsilon}B_0\hat{\phi}\right) r\, dr\, d\theta \tag{17}$$

$$= \frac{\lambda B_0}{2\pi}\hat{z}\int r dr\cdot 2\pi \tag{18}$$

$$\boxed{\vec{L}_i = \frac{1}{2}\lambda r^2 B_0\hat{z}} \tag{19}$$

Now we consider the final state. The $\vec{E}$ field remains unchanged by Gauss's law, but the $\vec{B}$ field is B_f as found in Eqn. (11). So citing Eqn. (19), the final angular momentum is

$$\begin{aligned}
\vec{L}_f &= \vec{L}_{mechanical} + \vec{L}_{EM} && (20)\\
&= I\omega_f \hat{z} + \frac{1}{2}\lambda r^2 B_f \hat{z} && (21)\\
&= I\omega_f \hat{z} + \frac{1}{2}\lambda r^2 \left(\frac{\mu_0}{2}\lambda\omega_f\right)\hat{z} && (22)\\
&= \omega_f \left(I + \frac{1}{4}\mu_0\lambda^2 r^2\right)\hat{z} && (23)\\
&= \frac{2\lambda r^2 B_0}{\mu_0\lambda^2 r^2 + 4I}\left(I + \frac{1}{4}\mu_0\lambda^2 r^2\right)\hat{z} && (24)\\
&= \frac{1}{4}\cdot\frac{2\lambda r^2 B_0}{(\mu_0\lambda^2 r^2/4 + I)}\left(I + \frac{1}{4}\mu_0\lambda^2 r^2\right)\hat{z} && (25)\\
&= \frac{1}{2}\lambda r^2 B_0 \hat{z} && (26)
\end{aligned}$$

i.e. $\boxed{\vec{L}_i = \vec{L}_f}$.