M10T2, M08T3

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Assume that, to escape from a metal, an electron must impinge from the interior onto the surface with enough momentum to overcome the confining potential that holds the electrons in the metal. Also assume that all electrons with such a momentum do escape. Calculate the flux (number per area per time) of electrons escaping from a metal with work function W (the barrier energy) at room temperature T. Treat the electrons as an ideal Fermi gas.

Note: The initial treatment is wrong, because I consider *energy* rather than the momentum perpendicular to the surface. Towards the end I consider momentum, and get a more correct answer. I keep the first part anyways because (1) it is illustrative of some of the concepts, and (2) it makes finding the correct answer much easier to explain having already explained it from an energy perspective. Also I already wrote it :p

We can convince ourselves that room temperature, $\approx 1/40$ eV, is in the low-temperature limit for an ideal fermi gas. This is true because $k_B T \ll E_F$ for a room-temperature metal.

Let's see this. At zero temperature, N equals the integral of the density of states from 0 to E_F, which allows us to implicitly solve for E_F. For a non-relativistic ideal fermi gas of electrons (spin-1/2),

$$g(E) = \frac{\sqrt{2} V m^{3/2} \sqrt{E}}{\pi^2 \hbar^3}$$

Thus

$$N = \int_0^{E_F} g(E) dE = \frac{2\sqrt{2} V m^{3/2} E_F^{3/2}}{3\pi^2 \hbar^3}$$

Solving for E_F, we have

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

N/V (number density) for atoms in metal is roughly $10^{28-29} m^{-3}$. Assuming metal has the same density as water, and assuming one electron per 2 amu (half protons, half neutrons, all electrons are part of fermi gas), I get $\approx 3 * 10^{29} m^{-3}$.

Plugging in numbers gives $E_F \approx 10eV$. This is much more than $k_B T$, so we are in the low-T fermi gas regime.

In this limit, $\frac{\partial \mu}{\partial T}\big|_{T=0} = 0$ (See David Tong Statistical Physics notes 3.6.2). So we can replace μ with it's value at $T = 0$, E_F. Note

$$n(E) = \frac{1}{e^{\beta(E-\mu)} + 1}$$

So back to the original problem at hand: How many electrons with energies between E and $E + dE$ escape from the metal per meter squared per second? Well this must be the number of electrons with energy E per volume, times the velocity of these electrons ($p/m = \sqrt{\frac{2E}{m}}$), times a geometrical factor of $\frac{1}{4}$ which covers the fact that the electrons have different directions (the origin of this factor of 1/4 is the same as the factor of 1/4 which comes from the calculation of the energy flux from blackbody radiation), times dE. Note that this works from dimensional analysis. The number of electrons with energy E per volume is $g(E)n(E)/V$.

Now we only have to determine the lowest E which electrons can escape the metal with (the upper limit on our energy will obviously be infinity). Since W, the work function, is defined as $-e\phi - E_F$ where $-e\phi$ is the electrostatic energy an electron needs to escape from the solid, then clear the energy an electron needs to escpae from the object is $W + E_F$. Thus the number of electrons escaping per meter squared per second is

$$\frac{1}{4V}\sqrt{\frac{2}{m}}\int_{W+E_F}^{\infty} g(E)n(E)\sqrt{E}dE$$

$$\frac{m}{2\pi^2\hbar^3}\int_{W+E_F}^{\infty} \frac{E}{e^{\beta(E-E_F)} + 1}dE$$

Changing variables from E to $E' = E - E_F$ we have

$$\frac{m}{2\pi^2\hbar^3}\int_{W}^{\infty} \frac{E' + E_F}{e^{\beta(E')} + 1}dE'$$

Now since $W \gg k_B T$, the denominator can be approximated in the numerator as $e^{-\beta E'}$. Thus we have

$$\frac{m}{2\pi^2\hbar^3}\int_{W}^{\infty} (E' + E_F)e^{-\beta(E')}dE'$$

Evaluating these integrals, we have

$$\frac{mk_B Te^{-\beta W}}{2\pi^2\hbar^3}(W + E_F + k_B T) \approx \frac{mk_B Te^{-\beta W}}{2\pi^2\hbar^3}(W + E_F)$$

Note that the dimensions are correct. However, this is different than Valentin's answer. Why? Well, the difference is due to the assumptions we made. I assumed that *every* electron with energy greater than $W + E_F$ which collided with the surface would escape. Valentin, on the other hand, assumed that every electron with $\frac{1}{2}mv_x^2 > W + E_F$ would escape.

Which is correct? Well, it looks like I fucked up here. The problem hints that the particles need to have enough *momentum* to overcome the confining potential. Physically, if there is a potential difference between the metal and the vacuum, then the energy from the direction perpendicular to the surface needs to overcome this potential, the velocity parallel is irrelevant to overcoming this potential. So let's alter the calculation slightly to fix this faulty assumption.

The density of states $g(v)$ is what? We can rederive the density of states as follows, using $E = \frac{\hbar^2\pi^2(n_x^2+n_y^2+n_z^2)}{2mL^2} = \frac{1}{2}mv^2$ and $n_i = \frac{mL}{\hbar\pi}v_i$.

$$\sum_{\vec{n}}(...) = \frac{g_s}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dn_x dn_y dn_z(...) = \frac{g_s}{8}V(\frac{m}{\hbar\pi})^3\int dv_x dv_y dv_z(...)$$

So $g(\vec{v}) = \frac{V}{4}(\frac{m}{\hbar\pi})^3$. So the amount leaving per second per meter squared is (following the same logic as before, removing the geometrical factor of 1/4, replacing p/m with v_x, integrating over all v_x and v_y, and integrating from $\frac{1}{2}mv_x^2 = W + E_F$ to infinity, and assuming $n(\vec{v}) \approx e^{-\beta(\frac{1}{2}m(v_x^2+v_y^2+v_z^2)-E_F)}$ because $\frac{1}{2}mv_x^2 >> k_BT$),

$$\frac{1}{4}(\frac{m}{\hbar\pi})^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\frac{1}{2}mv_x^2=W+E_F}^{\infty} dv_x dv_y dv_z(v_x e^{-\beta(\frac{1}{2}m(v_x^2+v_y^2+v_z^2)-E_F)})$$

Now the integrals over v_y and v_z are gaussian integrals, so these are each equal to $\sqrt{\frac{2\pi k_BT}{m}}$. The integral over v_x can be done most easily by changing variables to $E_x = \frac{1}{2}mv_x^2 - E_F$, $dE_x = mv_x dv_x$ so the integral becomes $\int_{E_x=W}^{\infty} dE_x e^{-\beta E_x}/m = \frac{k_BTe^{-\beta W}}{m}$. Putting everything together gives

$$\frac{m}{2\hbar^3\pi^2}e^{-\beta W}(k_BT)^2$$

This is essentially the same answer Valentin found - his answer differs from mine by a factor of $2\sqrt{2}$.