Prelims Solutions

Problem M10Q2

Valentin Skoutnev

1

$H(\lambda=0) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2 = \hbar\omega(a^\dagger a + \frac{1}{2})$ with $X = (\frac{\hbar}{2m\omega})^{\frac{1}{2}}(a+a^\dagger)$.

2

$E_o(\lambda) =< 0|H_1|0 >= \lambda < 0|X^4|0 >= \frac{\hbar^2\lambda}{4m^2\omega^2} < 0|aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + a^\dagger aaa^\dagger + aa^\dagger a^\dagger a + a^\dagger aa^\dagger a + a^\dagger a^\dagger aa|0 >$ since all other terms of the $< 0|(a+a^\dagger)^4|0 >$ expansion give zero. So using $a|n >= \sqrt{n}|n-1 >$ and $a^\dagger|n >= \sqrt{n+1}|n+1 >$:

$$E_o(\lambda) = \frac{\hbar^2\lambda}{4m^2\omega^2}(\sqrt{1}*\sqrt{2}*\sqrt{2}*\sqrt{1} + \sqrt{1}*\sqrt{1}*\sqrt{1}*\sqrt{1} + 0 + 0 + 0 + 0) = \frac{3\hbar^2\lambda}{4m^2\omega^2}$$

3

$|\psi_0(\lambda) >= |\psi_0(0) > + |\psi_1 >$ where $|\psi_1 >= \sum_{m=1}^{\infty} \frac{|m><m|\lambda X^4|0>}{E_0 - E_m} = \frac{|2><2|\lambda X^4|0>}{-2\hbar\omega} + \frac{|4><4|\lambda X^4|0>}{-4\hbar\omega}$ since $< 1|(a+a^\dagger)^4|0 >$ and $< 3|(a+a^\dagger)^4|0 >$ are zero.

$$< 2|(a+a^\dagger)^4|0 >=< 2|a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger aa^\dagger + a^\dagger aa^\dagger a^\dagger + aa^\dagger a^\dagger a^\dagger|0 >= 0 + \sqrt{2}\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{1} + \sqrt{3}\sqrt{3}\sqrt{2}\sqrt{1} = 6\sqrt{(2)}$$

$$< 4|(a+a^\dagger)^4|0 >=< 4|a^\dagger a^\dagger a^\dagger a^\dagger|0 >= \sqrt{(4!)} = 2\sqrt{(6)}$$

Hence, $|\psi_1 >= -\frac{\hbar\lambda}{4m^2\omega^3}(3\sqrt{2}|2 > + \frac{\sqrt{6}}{2}|4 >)$ Now,

$$< \psi_0(\lambda)|X^2|\psi_0(\lambda) >= \frac{\hbar}{2m\omega}(-\frac{\hbar\lambda}{4m^2\omega^3}\frac{\sqrt{6}}{2} < 4| - \frac{\hbar\lambda}{4m^2\omega^3}3\sqrt{2} < 2| + < 0|)(a+a^\dagger)^2(|0 > - \frac{\hbar\lambda}{4m^2\omega^3}3\sqrt{2}|2 > - \frac{\hbar\lambda}{4m^2\omega^3}\frac{\sqrt{6}}{2}|4 >)$$

Ugh,

$$< 0|(a+a^\dagger)^2|0 >=< 0|aa^\dagger|0 >= 1$$
$$< 2|(a+a^\dagger)^2|0 >=< 2|a^\dagger a^\dagger|0 >= \sqrt{2}$$
$$< 2|(a+a^\dagger)^2|2 >=< 2|a^\dagger a + aa^\dagger|2 >= 2 + 3 = 5$$
$$< 4|(a+a^\dagger)^2|0 >= 0$$
$$< 4|(a+a^\dagger)^2|2 >=< 4|a^\dagger a^\dagger|2 >= \sqrt{12}$$
$$< 4|(a+a^\dagger)^2|4 >=< 4|a^\dagger a + aa^\dagger|4 >= 4 + 5 = 9$$

Finally,

$$< \psi_0(\lambda)|X^2|\psi_0(\lambda) >= \frac{\hbar}{2m\omega}(0 - \frac{\hbar\lambda}{4m^2\omega^3}3*2 + 1 - \frac{\hbar\lambda}{4m^2\omega^3}\frac{\sqrt{6}}{2}(-\frac{\hbar\lambda}{4m^2\omega^3}3\sqrt{2})\sqrt{12} - \frac{\hbar\lambda}{4m^2\omega^3}3\sqrt{2}*(-\frac{\hbar\lambda}{4m^2\omega^3}3\sqrt{2})5 - \frac{\hbar\lambda}{4m^2\omega^3}3*2$$

$$-\frac{\hbar\lambda}{4m^2\omega^3}\frac{\sqrt{6}}{2}(-\frac{\hbar\lambda}{4m^2\omega^3}\frac{\sqrt{6}}{2})9 - \frac{\hbar\lambda}{4m^2\omega^3}3\sqrt{2}(-\frac{\hbar\lambda}{4m^2\omega^3}\frac{\sqrt{6}}{2})\sqrt{12} + 0 = -\frac{3\hbar^2\lambda}{2m^3\omega^4}$$

To first order in λ. Yea, I know, I didn't realize they asked for first order until too late.

4

Scale x as $x = \beta y$: $H = \frac{\hbar^2}{2m\beta^2}\frac{\partial^2}{\partial y^2}\psi + \lambda\beta^4 y^4\psi = E\psi$. We need $\frac{\hbar^2}{2m\beta^2} = \lambda\beta^4$ to make y dimensionless, so $\beta = (\frac{\hbar^2}{2m\lambda})^{\frac{1}{6}}$. Now $H = (\frac{\hbar^{\frac{4}{3}}\lambda^{\frac{1}{3}}}{(2m)^{\frac{2}{3}}})(\frac{\partial^2}{\partial y^2}\psi + y^4\psi) = E\psi$ so $\frac{\partial^2}{\partial y^2}\psi + y^4\psi = \epsilon\psi = (\frac{E(2m)^{\frac{2}{3}}}{\hbar^{\frac{4}{3}}\lambda^{\frac{1}{3}}})\psi \to E \propto \lambda^{\frac{1}{3}}$.