8.2 M10Q.3

a) Without loss of generalization, let $\mathbf{B} = B\hat{z}$. Then, the Hamiltonian is,

$$H = J\mathbf{S}_1 \cdot \mathbf{S}_2 - B\left(\alpha S_1^z + \beta S_2^z\right) = \frac{J\hbar^2}{4} \sum_{j=\{x,y,z\}} \sigma_1^j \otimes \sigma_2^j - \frac{B\hbar}{2}\left(\alpha \sigma_1^z \otimes \mathbb{I} + \beta \mathbb{I} \otimes \sigma_2^z\right)$$

where $\{\sigma^j\}$ are the Pauli matrices and $\mathbb{I}$ is the 2×2 identity matrix. In the basis $|S_1^z, S_2^z\rangle = |S_1^z\rangle \otimes |S_2^z\rangle$, i.e.

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

the Hamiltonian can be expressed as a 4×4 matrix given by,

$$H = \begin{pmatrix} \gamma - \lambda(\alpha+\beta) & 0 & 0 & 0 \\ 0 & -\gamma - \lambda(\alpha-\beta) & 2\gamma & 0 \\ 0 & 2\gamma & -\gamma + \lambda(\alpha-\beta) & 0 \\ 0 & 0 & 0 & \gamma + \lambda(\alpha+\beta) \end{pmatrix}$$

where $\gamma = J\hbar^2/4$ and $\lambda = B\hbar/2$. From the Hamiltonian, it is clear that $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states are eigenstates with the energies $\gamma \mp \lambda(\alpha+\beta)$. Let $\tilde{H}$ be the central 2×2 block of H. Then the eigenvalues of $\tilde{H}$ are,

$$(-\gamma - \omega - \mathcal{E})(-\gamma + \omega - \mathcal{E}) - 4\gamma^2 = 0 \Rightarrow \mathcal{E}_\pm = -\gamma \pm \sqrt{4\gamma^2 + \omega^2}$$

where $\omega = \lambda(\alpha - \beta)$. Therefore the energy eigenvalues for this system are,

$$\boxed{E = \frac{J\hbar^2}{4}\left[1 \pm \frac{2B}{J\hbar}(\alpha+\beta)\right], \quad -\frac{J\hbar^2}{4}\left(1 \pm 2\sqrt{1 + \frac{B^2}{J^2\hbar^2}(\alpha-\beta)^2}\right)}$$

b) From (a), we know that $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are already eigenstates. Therefore, we only need to find the eigenvectors of $\tilde{H}$, which follow

$$(-\gamma - \omega)c_1^\pm + 2\gamma c_2^\pm = (-\gamma \pm \sqrt{4\gamma^2 + \omega^2})c_1^\pm \Rightarrow c_2^\pm = \frac{\omega}{2\gamma}\left(1 \pm \sqrt{1 + \frac{4\gamma^2}{\omega^2}}\right) c_1^\pm$$

As a consistency check, setting $B = 0 \Rightarrow \omega = 0$ results in $c_2^\pm = \pm c_1^\pm$ which are just the mixed triplet state and the singlet state as expected. All in all, the eigenstates are,

$$\boxed{E(B;\alpha,\beta) = \begin{cases} \gamma - \lambda(\alpha+\beta), & |\uparrow\uparrow\rangle \\ \gamma + \lambda(\alpha+\beta), & |\downarrow\downarrow\rangle \\ -\gamma + \sqrt{4\gamma^2+\omega^2}, & N_+\left(|\uparrow\downarrow\rangle + c_+|\downarrow\uparrow\rangle\right) \\ -\gamma - \sqrt{4\gamma^2+\omega^2}, & N_-\left(|\uparrow\downarrow\rangle + c_-|\downarrow\uparrow\rangle\right) \end{cases}}$$

where

$$c_\pm = \frac{\omega}{2\gamma}\left(1 \pm \sqrt{1 + 4\gamma^2/\omega^2}\right), \quad N_\pm = \left(1 + c_\pm^2\right)^{-1/2}$$