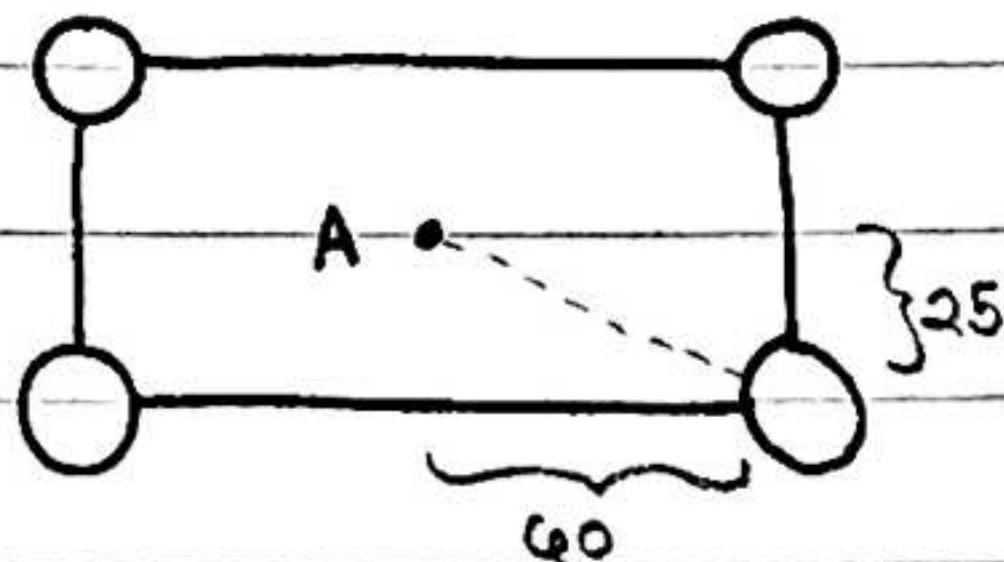


M10 M.3 - Rotating Space Station

23 November

4 pods, hollow spherical shells $m = 3 \cdot 10^4 \text{ kg}$ $r = 10 \text{ m}$ coupled together with rigid beams so centers are at corners of
 $dxh = 120 \text{ m} \times 50 \text{ m}$ rectangle

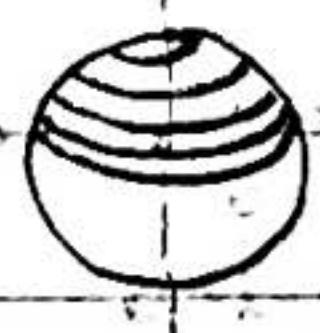
- a) compute moments of inertia I_1, I_2, I_3 relative to A for three principle axes $I_1 < I_2 < I_3$

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

need the moment of inertia of a spherical shell (sphere = $\frac{2}{3} MR^2$ shell = $\frac{2}{3} MR^2$)?

$$\text{ring} = mr^2 \quad dI = r^2 dm$$

$$dm = \frac{M}{A} dA \quad A = 4\pi r^2 \quad dA = 2\pi r \cdot R d\theta$$

want to sum disks from $r=0 \rightarrow R \rightarrow \theta$, $r = R \sin \theta$, $dr = R \cos \theta d\theta$

$$2 \int_0^R m r^2 dr$$

 m is mass of each ring

$$m = \rho dA = \rho 2\pi r R d\theta = \rho 2\pi r \frac{1}{\cos \theta} d\theta$$

$$= \frac{m}{4\pi R^2} 2\pi r \frac{1}{\cos \theta} d\theta$$

$$2 \int_0^R \frac{m^2 \pi}{4\pi R^2} r^3 \frac{1}{\cos \theta} d\theta$$

but $\cos \theta$ depends on r

$$\int_0^\pi m r^2 d\theta = \int_0^\pi \frac{m}{4\pi R^2} 2\pi r R r^2 d\theta = \frac{m}{2} \int_0^\pi R^2 \sin^3 \theta d\theta$$

$$= \frac{m R^2}{2} \int_0^\pi (1 - \cos^2 \theta) \sin \theta d\theta = \frac{m R^2}{2} \int_1^{-1} (1 - u^2) du$$

$$= \frac{-m R^2}{2} \left(u - \frac{1}{3} u^3 \right) \Big|_1^{-1} = \frac{-m R^2}{2} \left(-2 + \frac{1}{3} + \frac{1}{3} \right) = \frac{2}{3} m R^2 = I_p$$

$$-\frac{6}{3} + \frac{2}{3} = -\frac{4}{3} \quad * \text{gotta memorize}$$

vertically through A, \perp to plane of pods

$$I = 4(I_p + mr^2) = 4I_p + 4m(25^2 + 60^2) = 4I_p + 4m(5^2 5^2 + 12^2 5^2)$$

$$= \frac{8}{3} m r^2 + 100 m (5^2 + 12^2) = \frac{8}{3} m r^2 + 100 m 13^2 = \frac{8}{3} m r^2 + 100 m 169$$

$$= \left(\frac{8}{3} 100 + 16900 \right) 3 \cdot 10^4 \text{ kg m}^2 = (800 + 50700) \cdot 10^4 \text{ kg m}^2$$

$$\frac{16900}{16900} = 50700$$

$$= 5.15 \cdot 10^8 \text{ kg m}^2$$

$$I = 4(I_p + m(25)^2) = \left(\frac{8}{3} \cdot 100 + 4 \cdot 625\right) 3 \cdot 10^4 \text{ kg m}^2$$

$$= (800 + 2500 \cdot 3) \cdot 10^4 \text{ kg m}^2$$

$$= (8300) \cdot 10^4 \text{ kg m}^2 = \boxed{8.3 \cdot 10^7 \text{ kg m}^2}$$

$$I = 4(I_p + m(60)^2) = \left(\frac{8}{3} \cdot 100 + 4 \cdot 3600\right) 3 \cdot 10^4 \text{ kg m}^2$$

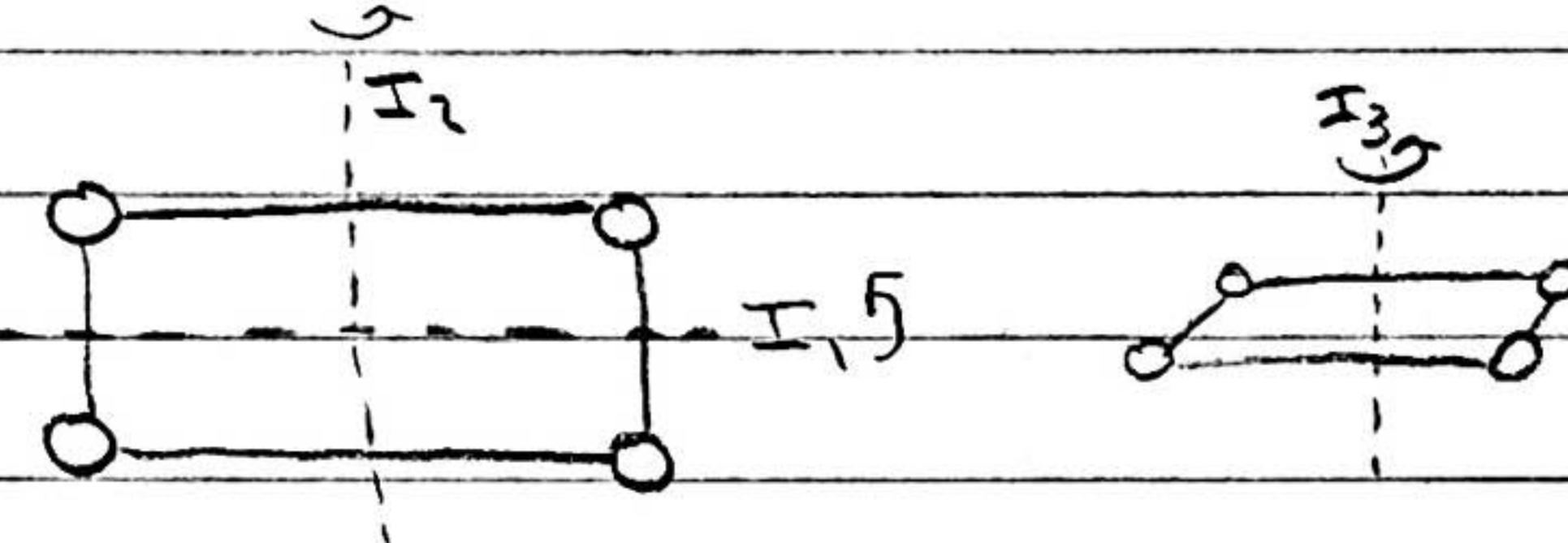
$$= (800 + 14400 \cdot 3) \cdot 10^4 = (44000 \cdot 10^4)$$

$$= 4.4 \cdot 10^8 \text{ kg m}^2$$

$$I_1 = 8.3 \cdot 10^7 \text{ kg m}^2$$

$$I_2 = 4.4 \cdot 10^8$$

$$I_3 = 5.15 \cdot 10^8$$



- b) hotel rotates about I_2 principal axis w/ freq f_g st astronaut at center of pod feels $\frac{1}{5} g$. Centrifugal force directed away from axis of rotation

what is value of f_g to generate required gravity?



$$F = ma = m\omega^2 r = m(2\pi f_g)^2 r$$

$$\text{centrifugal acceleration } a = \omega^2 r = \frac{v^2}{r}$$

$$v = r d\theta = \sqrt{x^2 + y^2} \quad a = r(4\pi^2 f^2) (-\cos(\theta), -\sin(\theta))$$

$$\text{position is } r\cos(\theta t), r\sin(\theta t) \quad v = r 2\pi f (-\sin(\theta t), \cos(\theta t))$$

$$f_g \text{ cycles/s}$$

$$r\cos(2\pi f_g t), r\sin(2\pi f_g t)$$

$$F = \frac{1}{5} g = mr(2\pi f_g)^2 = mr 4\pi^2 f_g^2$$

$$\frac{1}{5} g = \frac{1}{r 4\pi^2} = f_g^2$$

$$f_g = \sqrt{\frac{g}{20r\pi^2}}$$

$$\text{at } r = 60 \text{ m} \quad g = 10 \text{ m/s}$$

$$f_g = \frac{1}{\pi\sqrt{120}} \text{ /s}$$

$$f_g = \sqrt{\frac{10 \text{ m/s}}{20 \cdot 60 \text{ m/s}^2}} = \frac{1}{\pi\sqrt{120}} \text{ /s}$$

c) connecting beams slightly flexible, clamped small vibrations dissipate rotational KE as heat while conserving \vec{L}
 estimate characteristic time scale for exponential growth of ω_3 (ang. freq about I_3). Explain why I_2 is not stable wrt dissipation of rot kinetic energy

$$L_{\text{tot}} = L_{\text{rot}} + L_{\text{cm}}$$

outer equations

$$I_1 \frac{d\omega_1}{dt} = \omega_3 L_2 - \omega_2 L_3 = \omega_1 \omega_3 (I_2 - I_3)$$

$$I_2 \frac{d\omega_2}{dt} = \omega_1 L_3 - \omega_3 L_1 = \omega_1 \omega_3 (I_3 - I_1)$$

$$I_3 \frac{d\omega_3}{dt} = \omega_2 L_1 - \omega_1 L_2 = \omega_1 \omega_2 (I_1 - I_2)$$

should be able to get coupled 2nd order equations from outer equations
 and find characteristic exponential growth from that

$$I_1 \neq I_2 \neq I_3$$

$$L = I_2 \omega_2 \quad \text{this is conserved}$$

$$\text{max } \omega \text{ along } I_3 \text{ is } \omega_2 \frac{I_2}{I_3} = \omega_3 \quad \text{since } I_2 \omega_2 = L = I_3 \omega_3$$

but from kinetic energy conservation we have

$$KE = \frac{1}{2} I \omega^2 \text{ so initial value } \frac{1}{2} I_2 \omega_1^2 = \frac{1}{2} I_3 \omega_3^2 \text{ since some KE lost + heat}$$

$$\omega_3 \leq \omega_2 \sqrt{\frac{I_2}{I_3}} \quad \text{which is a more}$$

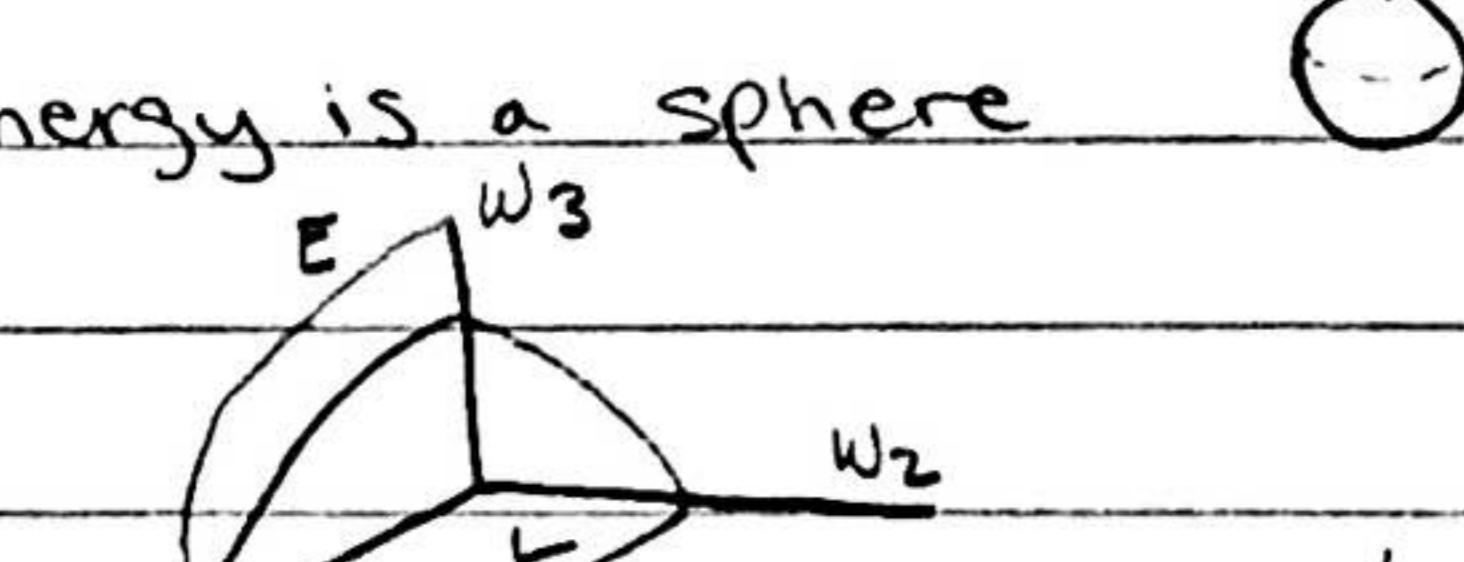
strict bound than the angular momentum conservation since $I_2 < I_3$

$$\omega_3 \leq \omega_2 \sqrt{\frac{I_2}{I_3}}$$

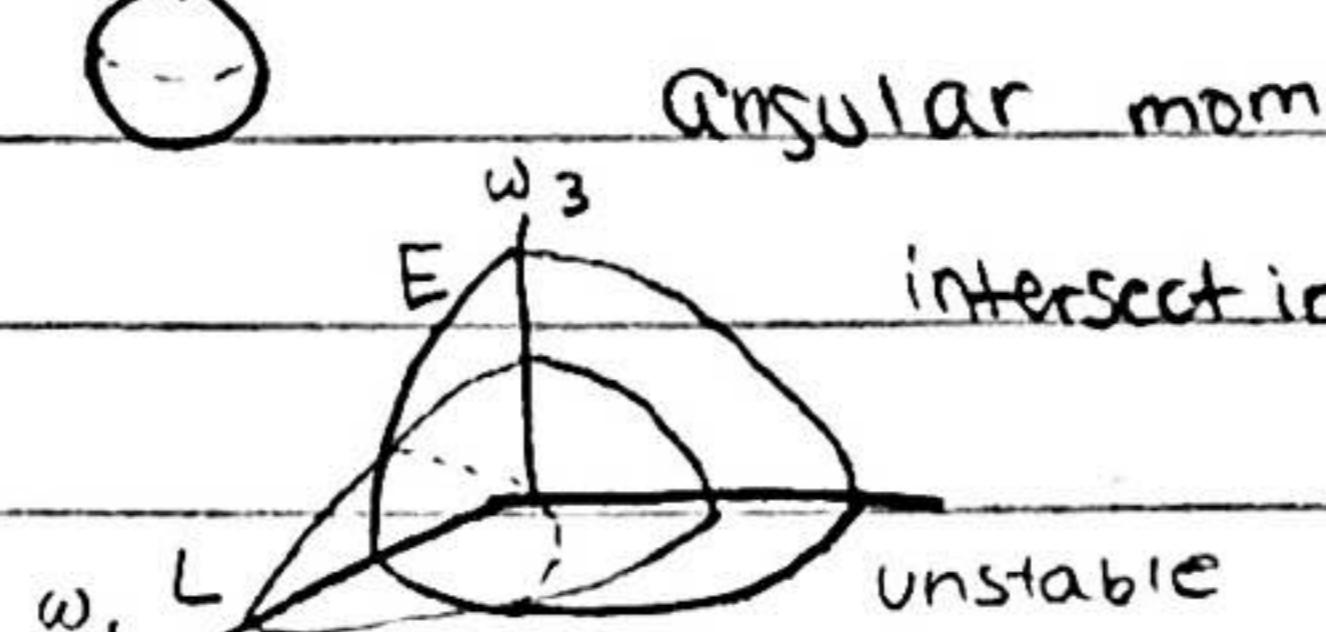
if on an intermediate axis due to allowed energy values you can have components along either I_1 or I_3 as well.

IF on I_1 or I_3 then due to energy and angular momentum constraints there is a small area of accessible phase space, so these are stable.

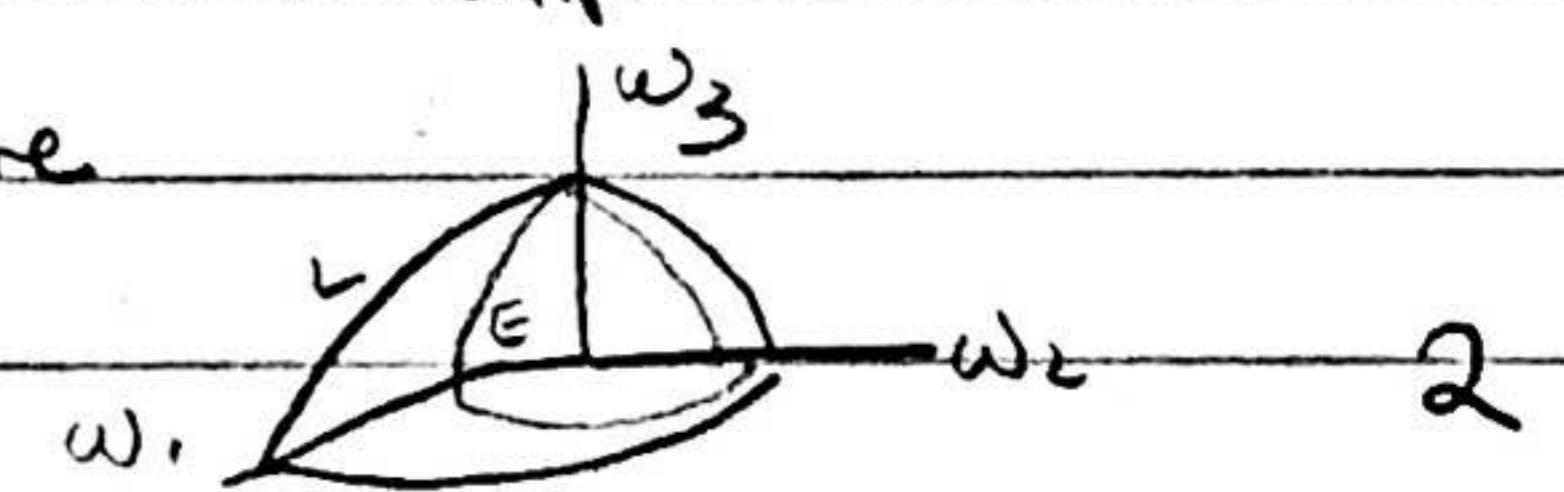
energy is a sphere



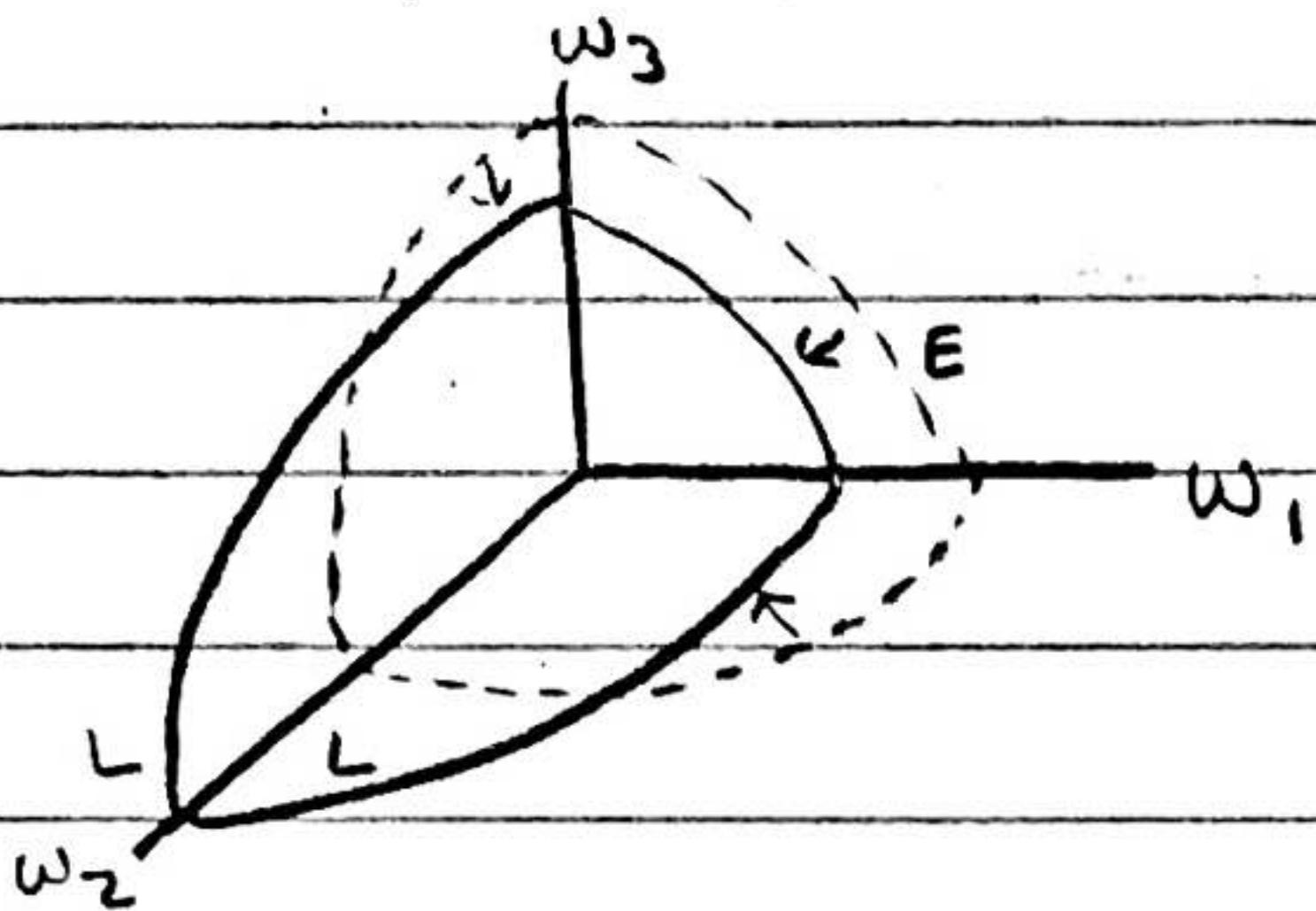
Angular momentum is an ellipsoid



intersection at two
unstable



If energy is dissipated, allowed energy sphere will eventually shrink inside the angular momentum ellipsoid, forcing the allowed areas of phase space to be at the ω_3 poles



as energy dissipates the accessible regions of phase space shrink

angular velocity vector is on the intersection of kinetic energy sphere and angular momentum ellipsoid

With Euler equations

$$I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_1 \omega_3$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2$$

Initially rotating around I_2 with ω_2

assume small ω_1 and ω_3 , so $\dot{\omega}_2$ is very small (≈ 0)

want to know time evolution of ω_3

$$I_3 \ddot{\omega}_3 = (I_1 - I_2) \omega_2 \dot{\omega}_1 + \underbrace{(I_1 - I_2) \omega_1 \omega_2}_{\approx 0} = (I_1 - I_2) \dot{\omega}_1 \omega_2$$

$$= (I_1 - I_2) \omega_2 \frac{1}{I_1} (I_2 - I_3) \omega_2 \omega_3$$

$$= \frac{1}{I_1} (I_1 - I_2) (I_2 - I_3) \omega_2^2 \omega_3$$

$$\ddot{\omega}_3 = \frac{1}{I_1 I_2} (I_1 - I_2) (I_2 - I_3) \omega_2^2 \omega_3 = K^2 \omega_3 \quad I_2 > I_1 \\ I_3 > I_1$$

$$\ddot{\omega}_3 = K^2 \omega_3$$

$$e^{K \omega_3} = \omega_3 (K)$$

exponential growth

K positive

characteristic time scale from $e^{x/\tau}$

$e^{K\omega_3}$ so characteristic time is $1/K = \tau$

$$\tau = \sqrt{I_1 I_3} ((I_1 - I_2)(I_2 - I_3) \omega_2^2)^{-1/2}$$

units $\frac{1}{\omega}$ = time as expected

a small disturbance is unstable and will cause exponential growth

unstable w/ or w/o kinetic energy dissipation assuming you start with a very slight instability. Potentially they mean for KE dissipation to provide this instability.