Problem M10E.2

(a) At large distances, the dominant contribution is magnetic dipole radiation. The magnetic moment of the loop varies as

$$\mathbf{m} = (2a)^2 I_0 \cos(\omega t)\,\hat{\mathbf{z}},$$

and so the vector potential has leading contribution

$$\begin{aligned}\mathbf{A}(t) &= -\frac{\mu_0}{4\pi r c}\hat{\mathbf{r}} \times \dot{\mathbf{m}}(t - r/c) \\ &= \frac{\mu_0 a^2 I_0 \omega}{\pi r c} \sin(\omega(t - r/c))\,(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) \\ &= -\frac{\mu_0 a^2 I_0 \omega}{\pi r c} \sin(\omega(t - r/c)) \sin\theta\,\hat{\boldsymbol{\phi}}.\end{aligned}$$

Thus the electric field is

$$\mathbf{E} = -\partial_t \mathbf{A} = -\frac{\mu_0 a^2 I_0 \omega^2}{\pi r c} \cos(\omega(t - r/c)) \sin\theta\,\hat{\boldsymbol{\phi}}.$$

(b) The Poynting vector is

$$\begin{aligned}\mathbf{S} &= \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \\ &= \frac{E^2}{\mu_0 c}\hat{\mathbf{r}} \\ &= \frac{\mu_0 a^4 I_0^2 \omega^4}{\pi^2 r^2 c^3} \cos^2(\omega(t - r/c)) \sin^2(\theta)\,\hat{\mathbf{r}}.\end{aligned}$$

Taking the time-average and integrating over all directions, we obtain the total radiated power

$$\mathbf{P} = \frac{4\mu_0 a^4 I_0^2 \omega^4}{3\pi c^3}$$

where we have used the integral

$$\oiint_{S^2} \sin^2\theta\,\mathrm{d}\Omega = \oiint_{S^2} (x^2 + y^2)\,\mathrm{d}\Omega = \frac{2}{3}\oiint_{S^2} (x^2 + y^2 + z^2)\,\mathrm{d}\Omega = \frac{8\pi}{3}.$$

(c) The conducting plane cancels the transverse component of $\mathbf{E}$, which is equivalent to an image loop located at $z = -2b$ with the opposite current.

Thus the dominant contribution is now magnetic quadrupole radiation. The magnetic quadrupole moment $\mathbf{Q}$ varies as $\cos(\omega t)$, and the vector potential is proportional to $\partial_t^2 \mathbf{Q} \propto \omega^2$. It follows that $E = -\partial_t \mathbf{A} \propto \omega^3$, and thus the Poynting vector is proportional to $\mathbf{E}^2 \propto \omega^6$.

Hence the total radiated power has an ω^6 dependence.