Prelims Solutions

Problem M09T3

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The gases will be in equilibrium when their chemical potentials are equal. The chemical potential for a 3D gas is $\mu = k_BTln(\frac{n_{3D}}{n_Q})$ where $n_{3D} = \frac{N_{3D}}{V} = \frac{P}{k_BT}$, $n_Q = (\frac{mk_BT}{\hbar^2 2\pi})^{\frac{3}{2}}$. The chemical potential for a 2D gas lives inside the expression for N_{2D}.

$$N_{2D} = \frac{1}{4}\int_0^\infty dn 2\pi n \frac{1}{e^{\beta(E-E_o-\mu)}-1}$$

where the energy levels are shifted down by E_o in the 2D gas to account for the binding. Using $E = \frac{\hbar^2\pi^2 n^2}{2mL^2} \rightarrow 2ndn = \frac{2mL^2}{\hbar^2\pi^2}dE$:

$$N_{2D} = \frac{\pi}{4}\frac{2mL^2}{\hbar^2\pi^2}\int_0^\infty dE\frac{1}{e^{\beta(E-E_o-\mu)}-1} = \frac{mL^2}{2\hbar^2\pi}(-k_BT)\int_0^\infty d(\beta E)\frac{1}{(-e^{-\beta(E_o+\mu)})e^{\beta E}+1}$$

Using the hint to evaluate the integral ($a = -e^{-\beta(E_o+\mu)}$):

$$\frac{N_{2D}}{L^2} = n_{2D} = \frac{m}{2\hbar^2\pi}(-k_BT)(ln(\frac{1}{-e^{-\beta E_o}e^{-\beta\mu}}) - ln(\frac{1}{1-e^{-\beta E_o}e^{-\beta\mu}})) = \frac{m}{2\hbar^2\pi}k_BTln(\frac{-e^{-\beta E_o}e^{-\beta\mu}}{1-e^{-\beta E_o}e^{-\beta\mu}})$$

Substituting μ for a 3D gas:

$$n_{2D} = \frac{m}{2\hbar^2\pi}k_BTln(\frac{1}{1-e^{\beta E_o}e^{\beta\mu}}) = \frac{m}{2\hbar^2\pi}k_BTln(\frac{1}{1-(\frac{P}{k_BTn_Q})e^{\beta E_o}})$$

which has the expected qualitative behavior, n_{2D} increases with increasing P and E_o.