Prelims Solutions

Problem M09T1

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1

We first need the partition function for a single particle:

$$Z_1 = \frac{1}{\hbar^5 4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp_x dp_y dp_z e^{-\frac{p_x^2+p_y^2+p_z^2}{2Mk_bT}} \int_{-\infty}^{\infty} dp_\theta e^{-\frac{p_\theta^2}{2Ik_bT}} *$$

$$* \int_0^{\pi} d\theta \int_{-\infty}^{\infty} dp_\phi e^{-\frac{p_\phi^2}{2Isin(\theta)^2 k_b T}} \int_0^{2\pi} d\phi e^{-\frac{\mu E cos(\theta)}{k_b T}} = \frac{V(2\pi m k_b T)^{\frac{3}{2}}(2\pi I k_b T)^{\frac{1}{2}}(2\pi I)^{\frac{1}{2}} 2k_b T sinh(\frac{\mu E}{T})}{2\hbar^5 \mu E}$$

For N identical particles $Z_N = Z_1^N/N!$. Thus, $F = -k_b T ln(Z_N)$.

2

The polarization is the average direction of $\vec{\mu}/V$, but $\vec{\mu}$ is going to be symmetric around the z axis since E is. Thus, we only need to find its average in the z direction. Notice

$$< \mu cos(\theta) > = -k_b T \frac{\partial ln(Z_N)}{\partial E}$$

So $P_N(T,V,E) = < \mu cos(\theta) > \frac{N}{V} = k_b T \frac{N\frac{\mu E}{k_b T}}{V sinh(\frac{\mu E}{k_b T})}(\frac{\frac{\mu}{k_b T} cosh(\frac{\mu E}{k_b T})}{\frac{\mu E}{k_b T}} - \frac{sinh(\frac{\mu E}{k_b T})}{\frac{\mu E^2}{k_b T}}) = \mu\frac{N}{V}(coth(\frac{\mu E}{k_b T}) - \frac{k_b T}{\mu E})$.

Expanding $coth(x) \approx 1/x + \frac{1}{3}x$ gives $P \approx \frac{N\mu}{3V}\frac{\mu E}{k_b T}$. Hence,

$$\epsilon = 1 + \frac{4\pi}{3}\frac{N\mu^2}{V k_b T}$$