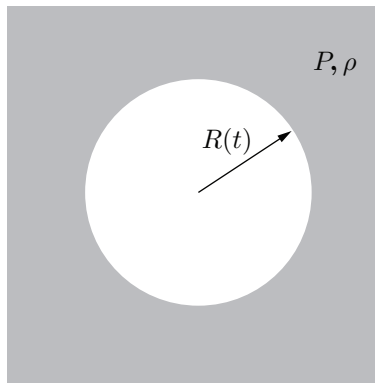


**M09M.1 - Bubble in an Incompressible Fluid (J07M.3, J94M.1)****Problem**

An ideal incompressible fluid of density  $\rho$  contains a bubble of radius  $R(t)$ . The fluid pressure is held constant at  $P$ . Take the bubble to be empty (no pressure within the bubble), neglect surface tension and gravity, and take the motion of the fluid as the bubble collapses to be radial.

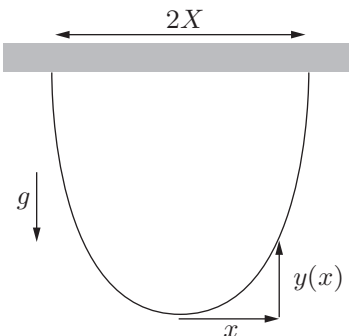


Find the rate of change  $dR/dt$  of the bubble radius after it has collapsed from rest by a factor of two in radius.

## M09M.2 - Shape of a Suspended Rope (M93M.3)

### Problem

An ideal perfectly flexible rope of length  $2L$  and fixed mass per unit length  $\mu$  is hanging at rest in a uniform gravitational field,  $g$ . The rope is held at its ends by supports at the same level and separated by distance  $2X$ .

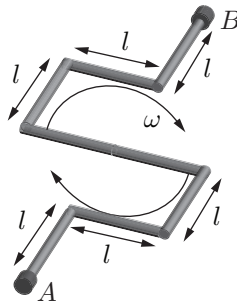


- Find the shape,  $y(x)$ , assumed by the rope.
- Find the tension,  $T_0$ , of the rope at its midpoint in the limit where  $X \ll L$ .

## M09M.3 - Automobile Crankshaft (J94M.3)

### Problem

As a model for an automobile crankshaft, consider a plane rigid body made of eight rods each of mass  $m$  and length  $l$ , as shown in the figure.



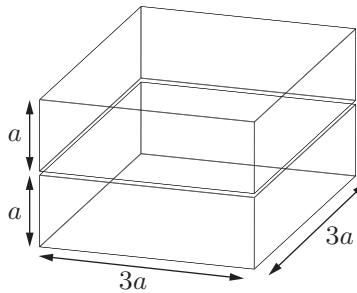
The bearings  $A$  and  $B$  constrain the crankshaft to rotate around a fixed axis at angular velocity  $\omega$ . Find the instantaneous force on one of the bearings. Ignore gravity and friction.

## M09E.1 - Cavity Dynamics (J94E.1)

### Problem

A cavity with perfectly conducting walls is formed by the planes  $x = y = z = 0, x = y = 3a, z = 2a$ . This cavity is sawed in two along the plane  $z = a$ . The cavity is excited in one of the modes which has the lowest frequency in view of the cut.

- Identify this mode and give its frequency.
- Calculate the average force tending to separate the two halves in terms of the maximum electric field  $E_0$ .



## M09E.2 - Pulsar (J94E.2)

### Problem

Approximate a pulsar to be a magnetized sphere of radius  $a$ . The pulsar has moment of inertia  $I$  and rotates about the  $z$  axis with angular velocity  $\omega$ . The magnetization,  $\mathcal{M}$ , is uniform and perpendicular to the  $z$  axis. In the sense that  $\omega a \ll c$ , the sphere is rotating slowly. As a result of the rotation the star is radiating.

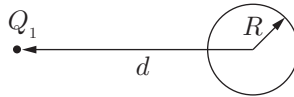
Assuming  $I$ ,  $\mathcal{M}$  and  $a$  to be constant, calculate the time rate of change of  $\omega$ .

## M09E.3 - Charge and Conducting Sphere (J94E.3)

### Problem

A point charge  $Q_1$  is located a distance  $d$  from the center of a thin, conducting spherical shell of radius  $R$  ( $d > R$ ).

- If the conducting sphere were temporarily grounded, what would be the magnitude  $Q'$  and distance from the origin  $d'$  of the image charge?
- Now the ground connection is removed and the conducting sphere is insulated from the ground, leaving the total net charge  $Q'$  distributed on the surface. If an additional amount of charge  $\Delta Q = Q - Q'$  is placed on the surface of the conducting sphere, how will the excess charge distribute itself on the surface?
- What must the net charge on this shell  $Q$  be so that there is no net force between the point charge  $Q_1$  and the shell?



## M09Q.1 - Hydrogen Beam (J94Q.1)

### Problem

A beam of hydrogen atoms is in the lowest hyperfine level of the  $1s$  state. The beam particles enter and pass through a region of space in which there is a uniform magnetic field of strength  $B = 100$  gauss. The particles move with a spread of velocities such that the time they spend in the magnetic field is distributed uniformly between 1 and 2 seconds. Treating the magnetic field effect as a small perturbation, compute to lowest order the fraction of atoms that are in the upper hyperfine level as they exit the magnetic field. A numerical answer is wanted.

Remarks:

The proton magnetic moment is negligibly small compared to that of the electron. For the electron, the Bohr magneton is  $e\hbar/2m_e c \approx 5.9 \times 10^{-9}$  eV/gauss.

The hyperfine wavelength (i.e. the wavelength of radiation emitted when the atom jumps from the upper to the lower hyperfine level) is  $\lambda = 21$  cm.

Planck's constant (divided by  $2\pi$ ) is:  $\hbar \approx 6.58 \times 10^{-16}$  eV s.

## M09Q.2 - Dynamics of Spin-1/2 Particles (J94Q.2)

### Problem

The spin dynamics of a spin-1/2 particle is governed by the Hamiltonian

$$H = \lambda\hbar\sigma_x.$$

At the initial time  $t = 0$  the spin is pointing up along the  $z$  axis. We are going to be concerned with measurements of the  $z$ -component of the spin at later times. Recall that in the standard representation

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We want to compare two situations:

- a) The  $z$ -component is measured at time  $\tau$ . What is the probability — call it  $P_I$  — that the spin will be pointing down?
- b) The  $z$ -component of spin is measured at an intermediate time  $\tau/2$ . This measurement is carried out by a compatriot who assures you that it was done but who doesn't tell you his finding. You carry out your own measurement at time  $\tau$ , i.e., at a time  $\tau/2$  after the intermediate measurement. What is the probability — call it  $P_{II}$  — that the spin is pointing down?



## M09Q.3 - Pion-Atom Scattering (J94Q.3)

### Problem

Consider the following 1-dimensional version of inelastic scattering of a projectile off an atom. In the 1- $d$  case one speaks of transmission-reflection coefficients rather than differential scattering cross sections.

Our 1- $d$  atom consists of an “electron” of mass  $m$ , position variable  $x_e$ , moving in a “nuclear” potential  $V(x_e)$ . Let  $u_0(x_e)$  and  $\epsilon_0$  be the (normalized) ground state eigenfunction and energy; let  $u_1(x_e)$  and  $\epsilon_1$  be the eigenfunction and energy of the first excited bound state. The projectile — a “pion” — has mass  $M$ , position variable  $x_p$ , and incident energy  $E = \hbar^2 k^2 / 2M$ . The projectile interacts with the electron with potential  $W(x_e - x_p)$ ; the projectile does not interact with the nucleus.

Consider the process in which the incident pion collides with the atom, which is initially in its ground state. The projectile is reflected in the backward direction with energy  $E' = \hbar^2 k'^2 / 2M$  and leaves the atom in the first excited state. You are given the energies of the atomic levels; so, given  $k$ , you know the momentum,  $-\hbar k'$ , of the reflected projectile. In addition to  $k$  and  $k'$  you are *given* the following (possibly complex) functions:

$$\tilde{V}(q) = \int dx e^{iqx} V(x), \quad \tilde{W}(q) = \int dx e^{iqx} W(x), \quad f(q) = \int dx u_1(x) e^{iqx} u_0(x).$$

For any of these functions you use, you must specify the argument  $q$  in terms of the given quantities, e.g.,  $k$  and  $k'$ .

Compute the reflection coefficient  $R$  for this process, in the lowest Born approximation.

Remark: The reflection coefficient is simply the transition rate to the final state under consideration, divided by the incident projectile flux.

## M09T.1 - Ideal Gas in an Electric Field (J94T.1)

### Problem

The Hamiltonian for a diatomic molecule with constant dipole moment  $\mu$  in a homogeneous electric field  $\vec{E} \equiv (0, 0, E)$  is

$$H = \frac{1}{2M}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2I}p_\theta^2 + \frac{1}{2I \sin^2 \theta}p_\phi^2 - \mu E \cos \theta$$

( $M$  = mass of molecule,  $I$  = moment of inertia, and  $(r, \theta, \phi)$  are polar coordinates).

Consider an ideal gas of  $N$  such *classical* molecules in a volume  $V$ , using *Boltzmann statistics*.

- Compute the free energy  $F_N(T, V, E)$ .
- Compute the dipole moment per volume (“polarization”),  $P_N(T, V, E)$ , of the gas and evaluate the dielectric constant  $\epsilon$  in the limit  $\mu E \ll k_B T$ .

(Recall:  $\epsilon E = E + 4\pi P$ .)

## M09T.2 - Particles on a Line (J94T.2)

### Problem

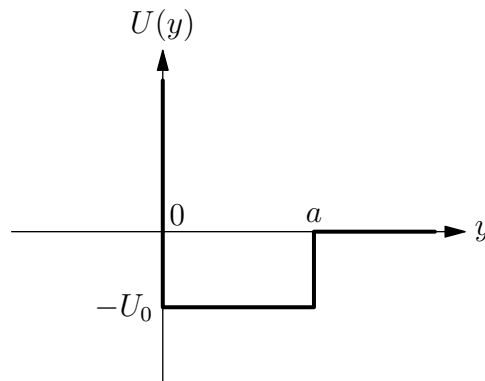
Consider a system of  $N$  classical particles on a line with Hamiltonian

$$H = \frac{p_1^2}{2m} + U(x_1) + \sum_{i=2}^N \frac{p_i^2}{2m} + U(x_i - x_{i-1}).$$

The potential between neighboring particles has the form:

$$U(y) = \begin{cases} +\infty, & \text{if } y < 0; \\ -U_0, & \text{if } 0 \leq y \leq a; \\ 0, & \text{if } a < y. \end{cases}$$

( $U_0 > 0, a > 0$ )



In addition, a constant force  $f$  is applied to the rightmost particle  $i = N$ .

- Compute the mean length,  $\langle x_n \rangle$ , of the system as a function of  $N, T$  and  $f$ .
- Obtain the high and low temperature limits of the result from part a).

## M09T.3 - 2D and 3D Bosons (J94T.3)

### Problem

Consider a 3-dimensional gas of (spinless, non-relativistic) bosons at pressure  $P$  and temperature  $T$ . The bosons can be absorbed onto a (2-dimensional) surface layer, where they are bound with energy  $-\epsilon_0 < 0$ , but retain their translational degrees of freedom in 2 dimensions. The (ideal) 3D gas is in equilibrium with the (ideal) 2D absorbed gas.

Treating the 3D gas classically, but the 2D gas quantum mechanically, compute the surface density in the layer as a function of  $P$  and  $T$ .

(You may need:  $\int \frac{dx}{ae^x+1} = \ln \frac{e^x}{1+ae^x}$ .)