SM M08 #1

a) $m \frac{d\vec{v}}{dt} = \vec{F}(t) - 6\pi\eta a \vec{v}(t)$

$$\frac{d\vec{v}}{dt} = \frac{1}{m} \vec{F}(t) - \frac{6\pi\eta a}{m} \vec{v}(t)$$

$$\int \frac{d\vec{v}}{dt} = v(t+T) - v(t) = -\gamma v(t) T + \frac{1}{m} \int_t^T F(t')dt'$$

$$\langle v(0) v(t+T) \rangle - \langle v(0) v(t) \rangle = -\gamma T \langle v(0) \cdot v(t) \rangle + \frac{1}{m} \langle v(0) \int_t^T F(t')dt' \rangle$$

last term = 0 b/c quickly varying

$$\frac{\langle v(0) v(t+T) \rangle - \langle v(0) v(t) \rangle}{T} = -\gamma \langle v(0) v(t) \rangle$$

$$\frac{d}{dt} \langle v(0) v(t) \rangle = -\gamma \langle v(0) v(t) \rangle$$

$$\langle v(0) v(t) \rangle = C_0 e^{-\gamma t} \qquad C_0 = \langle v(0)^2 \rangle$$

$$\langle v(0) \cdot v(t) \rangle = \langle v(0)^2 \rangle e^{-\gamma t}$$

$$\frac{1}{2} m \langle v(0)^2 \rangle \approx \frac{1}{2} kT$$

$$\langle \vec{v}(0) \cdot \vec{v}(t) \rangle \approx \frac{kT}{m} e^{-\gamma t} \qquad \gamma = \frac{6\pi\eta a}{m}$$

b) $\langle (\vec{r}(t) - \vec{r}(0))^2 \rangle = 2 \int_0^t dt' (t - t') \langle v(0) \cdot v(t') \rangle$

$$= 2 \frac{kT}{m} \left\{ t \int_0^t e^{-\gamma t'} dt' - \int_0^t t' e^{-\gamma t'} dt' \right.$$

$\int u\,dv = uv - \int v\,du \qquad u = t, \quad dv = e^{-\gamma t}$

$$\int_0^t t e^{-\gamma t} dt = -t \frac{1}{\gamma} e^{-\gamma t} + \frac{1}{\gamma} \int e^{-\gamma t} dt$$

$$= \frac{-t'}{\gamma} e^{-\gamma t'} - \frac{1}{\gamma^2} e^{-\gamma t'} \Big|_0^t$$

$$= \frac{-t}{\gamma} e^{-\gamma t} - \frac{1}{\gamma^2} e^{-\gamma t} + \frac{1}{\gamma^2}$$

$$\langle \cdots \rangle = 2 \frac{kT}{m} \left[\frac{-t}{\gamma} \left[e^{-\gamma t} - 1 \right] - \left[\frac{-t}{\gamma} e^{-\gamma t} - \frac{1}{\gamma^2} e^{-\gamma t} + \frac{1}{\gamma^2} \right] \right]$$

$$= \frac{2kT}{m} \left[\frac{t}{\gamma} + \frac{1}{\gamma^2} e^{-\gamma t} - \frac{1}{\gamma^2} \right]$$

$$\boxed{\langle (\vec{r}(t) - \vec{r}(0))^2 \rangle = \frac{2kT}{m\gamma} \left[t - \frac{1}{\gamma} \left(1 - e^{-\gamma t} \right) \right]}$$

$$\langle x^2 \rangle \rightarrow \frac{2kT}{m\gamma} t \quad \text{for large } t$$

T, m, γ are known and $\langle x^2 \rangle$ can be measured experimentally. Thus k can be determined.