M08Q.3 DYNAMICS OF SPIN

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1. Problem

A spin of s = 1/2 has its z-component "up" at time t = 0. The dynamics of the spin are given by the Hamiltonian:

$$H = \lambda\hbar\sigma_x,$$

where σ_x is the usual Pauli matrix for a spin-1/2.

1.1. **(a.)** If the z-component of the spin is measure at time t = τ, what are the probabilities of each possible result of this measurement?

We define our states so that $\begin{bmatrix}1\\0\end{bmatrix}$ is the z-spin up state, and $\begin{bmatrix}0\\1\end{bmatrix}$ is the z-spin down state. Then the Pauli matrix has the form:

$$\sigma_x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

This gives a Hamiltonian:

$$H = \begin{bmatrix}0 & \lambda\hbar\\ \lambda\hbar & 0\end{bmatrix}$$

The eigenvalues of this matrix are clearly $\pm\lambda\hbar$, with corresponding eigenstates:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \pm 1\end{bmatrix}.$$

Now, we are given the initial state is z-spin up, which we form from the eigenstates, and by adding the time dependence get the full time-dependent wavefunction:

$$\Psi = \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix}e^{i\lambda t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-i\lambda t}\right)$$

Now we need only dot this with the transpose of z-spin up at time τ, or explicitly, perform $\begin{bmatrix}1 & 0\end{bmatrix} * \Psi(t=\tau)$, the result of which will be the probability amplitude of measuring spin up:

$$\begin{bmatrix}1 & 0\end{bmatrix} * \Psi(t=\tau) = \frac{1}{2}(e^{i\lambda\tau} + e^{-i\lambda\tau}) = \cos(\lambda\tau)$$

The probability is the square of this amplitude, so the probability of measuring z-spin up is $\cos^2(\lambda\tau)$. We immediately recognize that the probability of measuring spin down must be $\sin^2(\lambda\tau)$ for normalization (spin up and down are the only outcomes possible).

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1.2. **(b.)** Now the same system is measured twice, once at time $\tau/2$ and again at time τ. The Hamiltonian still governs the spin's dynamics betwixt[1] measurements.

Since we do not know the result of the spin measurement at $t = \tau/2$, we take the superposition of z-spin up and z-spin down as the state at $t = \tau/2$, weighted by the probabilities found in part a (adjusted for the time difference):

$$\begin{aligned}\Psi(t=\tau/2) &= \cos\left(\lambda\frac{\tau}{2}\right)\begin{bmatrix}1\\0\end{bmatrix} + \sin\left(\lambda\frac{\tau}{2}\right)\begin{bmatrix}0\\1\end{bmatrix}\\ &= \cos\left(\lambda\frac{\tau}{2}\right)\frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right) + \sin\left(\lambda\frac{\tau}{2}\right)\frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\right)\end{aligned}$$

Then the time-dependent wavefunction is:

$$\Psi(t) = \frac{\cos\left(\lambda\frac{\tau}{2}\right) + \sin\left(\lambda\frac{\tau}{2}\right)}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{i\lambda t} + \frac{\cos\left(\lambda\frac{\tau}{2}\right) - \sin\left(\lambda\frac{\tau}{2}\right)}{2}\begin{bmatrix}1\\-1\end{bmatrix}e^{-i\lambda t}$$

Now we repeat the process from the end of part a, dotting the transpose of the z-spin up with the wavefunction at $t = \tau$.

$$\begin{aligned}\begin{bmatrix}1 & 0\end{bmatrix} * \Psi(t=\tau) &= \frac{\cos\left(\lambda\frac{\tau}{2}\right)}{2}(e^{i\lambda\tau} + e^{-i\lambda\tau}) + \frac{\sin\left(\lambda\frac{\tau}{2}\right)}{2}(e^{i\lambda\tau} - e^{-i\lambda\tau})\\ &= \cos\left(\lambda\frac{\tau}{2}\right)\cos(\lambda\tau) + i\sin\left(\lambda\frac{\tau}{2}\right)\sin(\lambda\tau)\end{aligned}$$

So the probability (square of this amplitude) of measuring z-spin up is:

$$\cos^2\left(\lambda\frac{\tau}{2}\right)\cos^2(\lambda\tau) + \sin^2\left(\lambda\frac{\tau}{2}\right)\sin^2(\lambda\tau).$$

For normalization, the probability of measuring z-spin down must be:

$$\cos^2\left(\lambda\frac{\tau}{2}\right)\sin^2(\lambda\tau) + \sin^2\left(\lambda\frac{\tau}{2}\right)\cos^2(\lambda\tau).$$

[1]Betwixt is an archaic synonym for between. Since there is no confusion in its replacement, I characteristically use betwixt when writing up physics problems.