Ladder Operator

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PROBLEM

The dynamics of a system is characterized by the Hamiltonian

$$H = a^\dagger a + \frac{1}{2}\,, \quad \left[a, a^\dagger\right] = 1$$

a) Show that the ground state of this system, $|0\rangle$, satisfies $a|0\rangle = 0$.

b) Consider the state

$$|\alpha\rangle = \mathcal{N} e^{\alpha a - \alpha^* a^\dagger} |0\rangle\,,$$

where $\mathcal{N}$ is some normalization constant. Show that $a|\alpha\rangle = \alpha|\alpha\rangle$. Find $\mathcal{N}$.

c) Consider the change of variables

$$a = \frac{1}{\sqrt{2}}(q + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$$

Derive and interpret the Hamiltonian in this set of new variables.

d) Calculate $\langle\alpha|q|\alpha\rangle$. Describe the time dependence of $\langle\alpha|q|\alpha\rangle$.

SOLUTION TO PART A

Obviously operator $a^\dagger a$ is Hermitian, then it has real eigenvalues λ with eigenstates $|\lambda\rangle$:

$$a^\dagger a|\lambda\rangle = \lambda|\lambda\rangle$$

By the commutation relation $\left[a, a^\dagger\right] = 1$ we get the following result

$$a^\dagger a a|\lambda\rangle = (\lambda - 1)a|\lambda\rangle$$

That means $a|\lambda\rangle$ is an eigenstate of $a^\dagger a$ with eigenvalue $\lambda - 1$. Assume the eigenstates are normalized by $\langle\lambda|\lambda\rangle = 1$, we then have the magnitude of the state $a|\lambda\rangle$:

$$||a|\lambda\rangle|| = \langle\lambda|a^\dagger a|\lambda\rangle = \lambda > 0$$

That tells us $a|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle$. Now we assume that λ is not an integer, then we can multiply by operator a to the state $|\lambda\rangle$ again and again. Every time you use the operator a, the eigenvalue of the state is reduced by one, and you can always get a negative λ. But λ is also the magnitude of a state which means it must be non-negative. So the only way to get a self-consist theory is to set λ an integer. When $\lambda = 0$, we can get

$$a|0\rangle = 0\,.$$

SOLUTION TO PART B

The well-known BCH formula is

$$e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]}$$

Now we set $A = \alpha a^\dagger$ and $B = -\alpha^* a$, we can derive

$$e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}$$

Now multiply this operator to the vacuume state $|0\rangle$, we get

$$\begin{aligned}
\mathcal{N} e^{\alpha a^\dagger - \alpha^* a}|0\rangle &= \mathcal{N} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle \\
&= \mathcal{N} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger}|0\rangle \\
&= \mathcal{N} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(a^\dagger\right)^n |0\rangle \\
&= \mathcal{N} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\end{aligned}$$

From the normalization condition $\langle\alpha|\alpha\rangle = 1$, we find that $\mathcal{N} = 1$. Then we can check that $|\alpha\rangle$ is the eigenstate of operator a:

$$\begin{aligned}
a\left(e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}|n\rangle}\right) &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle \\
&= \alpha e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}} |n\rangle \\
&= \alpha e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
&= \alpha|\alpha\rangle
\end{aligned}$$

SOLUTION TO PART C

$$H = a^\dagger a + \frac{1}{2} = \frac{1}{2}(q + ip)(q - ip) + \frac{1}{2} = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{i}{2}[q,p] + \frac{1}{2}\,,$$

Next we can use $\left[a, a^\dagger\right] = 1$ to derive the commutation relation for q and p, and we get

$$[q,p] = \frac{1}{2i}\left(\left[a^\dagger, a\right] - \left[a, a^\dagger\right]\right) = i\,,$$

Then the Hamiltonian becomes

$$H = \frac{1}{2}p^2 + \frac{1}{2}p^2$$

SOLUTION TO PART D

It is easy to get the average value of q under the state $|\alpha$:

$$\langle\alpha|q|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\left(a+a^\dagger\right)|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\alpha+\alpha^*\right) = \sqrt{2}\mathrm{Re}\,\alpha\,.$$

Then we need to derive the time evolution of this state. The Hamiltonian is given by $H = a^\dagger a + \frac{1}{2}$, so the unitary time evolution operator is

$$U(t) = e^{-i\frac{t}{2}}e^{-ia^\dagger at}\,,$$

So the qunatum state at time t is given by

$$\begin{aligned}
|\alpha(t)\rangle &= U(t)|\alpha\rangle \\
&= e^{-i\left(a^\dagger a+1/2\right)t}e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\
&= e^{-i\frac{t}{2}}e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-int}|n\rangle \\
&= e^{-i\frac{t}{2}}e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha e^{-it})^n}{\sqrt{n!}}|n\rangle \\
&= e^{-i\frac{t}{2}}|\alpha e^{-it}\rangle
\end{aligned}$$

That means under the unitary time evolution, the coherent state $|\alpha\rangle$ is still a coherent state, but the eigenvalue is changing under the following rule

$$\alpha(t) = \alpha(0)e^{-it}$$

Since the average value of q is proportional to the real part of eigenvalue α, we knwo that it is oscillating with frequecy $\omega = 1$:

$$\langle\alpha(t)|q|\alpha(t)\rangle = \sqrt{2}|\alpha|\cos(t+\varphi)$$

in which $|\alpha|$ and φ are determined by the initial condition.