M08M.2 Solution

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The equations of motions can be found (through trial and error if not other methods) to be:

$$\begin{aligned} m_1\ddot{x_1} =& -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x_2} =& -k_3x_2 - k_2(x_2 - x_1) \end{aligned}$$

We then make the ansatz that the solution will be of the form

$$\begin{aligned} x_1 =& Ae^{i\alpha t} \\ x_2 =& Be^{i\alpha t} \end{aligned}$$

Plugging this back into the equations of motion we get

$$\begin{aligned} m_1(-\alpha^2 A) =& -k_1A + k_2(B - A) \\ m_2(-\alpha^2 B) =& -k_3B - k_2(B - A) \end{aligned}$$

Rewriting this into matrix form we get

$$\begin{bmatrix} -m_1\alpha^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\alpha^2 + k_2 + k_3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

For this to have a nontrivial solution the matrix must not be invertible (or in other words the determinant must be 0)

$$\Rightarrow 0 = (-m_1\alpha^2 + k_1 + k_2)(-m_2\alpha^2 + k_2 + k_3) - k_2^2$$

$$0 = m_1m_2\alpha^4 - (m_1(k_2 + k_3) + m_2(k_1 + k_2))\alpha^2 + (k_1 + k_2)(k_2 + k_3) - k_2^2$$

Solving the quadratic I get that

$$\begin{aligned} \alpha_\pm^2 = \frac{1}{2m_1m_2}\Bigg(& m_1(k_3 + k_2) + m_2(k_1 + k_2) \\ &\pm \sqrt{[m_1(k_2 + k_3) + m_2(k_1 + k_2)]^2 - 4m_1m_2\big((k_1 + k_2)(k_2 + k_3) + k_2^2\big)}\Bigg) \end{aligned}$$

Plugging this back into equation (1) we can solve for the relationship between A and B. Using the top row I get

$$\frac{B}{A} = -\frac{1}{2m_1k_2}\Bigg(m_1(k_3+k_2)+m_2(k_1+k_2)$$
$$\pm\sqrt{[m_1(k_2+k_3)+m_2(k_1+k_2)]^2-4m_1m_2\big((k_1+k_2)(k_2+k_3)+k_2^2\big)}\Bigg)+\frac{k_1}{k_2}+1$$

The bottom row should give the same answer. Let me call this ratio $\gamma_\pm$. Thus the solution to the equations of motion for our system is

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\\gamma_+\end{bmatrix}e^{-i\alpha_+t}+\begin{bmatrix}1\\\gamma_+\end{bmatrix}e^{+i\alpha_+t}+\begin{bmatrix}1\\\gamma_-\end{bmatrix}e^{-i\alpha_-t}+\begin{bmatrix}1\\\gamma_-\end{bmatrix}e^{+i\alpha_-t}$$

The normal modes are each of the terms of this solution.