M08M.1 Solution

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a) To find the equilibrium position we need an equation of motion in θ which should tell us we need Lagrangian (unless you wanna go the Newtonian route).

$$\begin{aligned}
T =& \frac{1}{2} I_\omega \omega^2 + \frac{1}{2} I_\theta \dot{\theta}^2 + \frac{1}{2} I_{ring} \omega^2 \\
V =& -ma\cos\theta
\end{aligned}$$

Where $I_\omega = m\left(a\sin\theta\right)^2$ is the bead rotating around the vertical axis, $I_\theta = ma^2$ is the bead rotating along the hoop, and $I_{ring} = \frac{1}{2}Ma^2$ is the rotation of the hoop itself.

$$\begin{aligned}
\mathcal{L} =& \frac{1}{4} Ma^2\omega^2 + \frac{1}{2} m\left(a\sin\theta\right)^2\omega^2 + \frac{1}{2} ma^2\dot{\theta} + ma\cos\theta \\
\frac{\partial \mathcal{L}}{\partial \theta} =& ma^2 \sin\theta\cos\theta\omega^2 - a\sin\theta mg \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) =& ma^2\ddot{\theta} \\
\ddot{\theta} =& \sin\theta\cos\theta\omega^2 - \sin\theta\frac{g}{a}
\end{aligned}$$

The equilibrium position is where $\ddot{\theta} = 0$.

$$0 = \sin\theta\left(\cos\theta\omega^2 - \frac{g}{a}\right)$$

The $\sin\theta$ tells us that two equilibrium positions are $\theta = 0, \pi$

$$\begin{aligned}
0 =& \cos\theta\omega^2 - \frac{g}{a} \\
\cos\theta =& \frac{g}{\omega^2 a}
\end{aligned}$$

Thus $\theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 a}\right), 0, \pi$. It should be noted that if $\omega^2 < \frac{g}{a}$ then the ω dependent solution no longer exists since $\cos\theta \leq 1$. Furthermore $\theta = \pi$ is an unstable equilibrium which can be seen by plugging $\theta = \pi + \delta$ into the equation of motion

$$0 = \ddot{\delta} - \delta\left(\omega^2 + \frac{g}{a}\right)$$

thus small perturbations to $\theta = \pi$ does not have oscillatory behavior. For $\omega^2 > \frac{g}{a}$ then $\theta = 0$ is also unstable. Expanding the equation of motion for theta around small angles we get

$$0 = \ddot{\theta} + \theta\left(\frac{g}{a} - \omega^2\right)$$

Which is only oscillatory if $\omega^2 < \frac{g}{a}$. Thus the only stable equilibrium position is $\theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 a}\right)$

b) Based on what I was saying at the end of part a), by assuming $\omega^2 > \frac{g}{a}$ we can ignore the $\theta_0 = 0, \pi$ solutions since it needs to undergo oscillations after being perturbed. Now I say that in order for ω to be assumed to be constant $\frac{d\omega}{dt}/\omega << \omega$ or in other words the change of ω in one period of the hoop must a negligible fraction of ω. Using the Lagrangian from part a) to find the equation of motion in ϕ where $\dot{\phi} = \omega$, I get the following

$$\frac{1}{2}Ma^2\ddot{\phi} + ma^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + m(a\sin\theta)^2\ddot{\phi} = 0$$
$$\Rightarrow \left(\frac{1}{2}Ma^2 + m(a\sin\theta)^2\right)\frac{\dot{\omega}}{\omega} + ma^2\sin\theta\cos\theta\dot{\theta} = 0$$

Solving for the magnitude of $\frac{\dot{\omega}}{\omega}$ and using the condition stated above I get the equation

$$\frac{m\sin\theta\cos\theta}{\frac{1}{2}M + m\sin^2\theta}\dot{\theta} << \omega$$

this is somewhat unsatisfying since we don't really know what $\dot{\theta}$ is yet. This will be answered in part c) however once we find the differential equation for small oscillations

c) Using the equation of motion found in part a) I put in the transformation $\theta \to \theta_0 + \Delta\theta$

$$\ddot{\Delta\theta} = \sin(\theta_0 + \Delta\theta)\left(\omega^2\cos(\theta_0 + \Delta\theta) - \frac{g}{a}\right)$$

Once again using angle addition formulas I get

$$\ddot{\Delta\theta} = (\sin(\theta_0)\cos(\Delta\theta) + \sin(\Delta\theta)\cos(\theta_0)) \left(\omega^2 \left(\cos(\theta_0)\cos(\Delta\theta) - \sin(\Delta\theta)\sin(\theta_0)\right) - \frac{g}{a}\right)$$
$$\ddot{\Delta\theta} = \left(\sin\theta_0 + \Delta\theta\frac{g}{\omega^2 a}\right)\left(\omega^2\left(\frac{g}{\omega^2 a} - \Delta\theta\sin\theta_0\right) - \frac{g}{a}\right)$$
$$\ddot{\Delta\theta} = \left(\sin\theta_0 + \Delta\theta\frac{g}{\omega^2 a}\right)(-\omega^2\Delta\theta\sin\theta_0)$$

Dropping the terms proportional to $\Delta\theta^2$ we get

$$\ddot{\Delta\theta} = -\,\omega^2\Delta\theta\sin^2\theta_0$$

Thus the frequency of small oscillations ω_θ is

$$\omega_\theta = \omega\sin\theta_0 = \omega\sqrt{1 - \left(\frac{g}{a\omega^2}\right)^2}$$

This also goes back to part b). Since now we have a complete equation, $\theta = \theta_0 + \Delta\theta_0\cos(\omega_\theta t)$, we can plug this in and get a condition that depends only on g, ω, m, M, a and one initial condition $\Delta\theta$ which are the given variables in the problem. This is left as an exercise to the reader since this gets very messy very quickly