May 2008 #2 (EM)

a. $\vec{E}_a = \frac{e}{c}\left[\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{ret}$ $\qquad \vec{B} = [\hat{n}\times\vec{E}]_{ret}$

nonrelativistic: $\beta\to 0$,

$$\vec{E}_a = \frac{e}{c}\left(\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R}\right)_{ret}$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\vec{E}\times(\hat{n}\times\vec{E}) = \frac{c}{4\pi}\hat{n}|\vec{E}|^2$$

$$\frac{dP}{d\Omega} = \vec{S}\cdot\hat{n}R^2 = \frac{c}{4\pi}|R\vec{E}|^2 = \frac{c}{4\pi}\cdot\frac{e^2}{c^2}\cdot\left|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right|^2 = \frac{e^2}{4\pi c}\dot{\beta}^2\sin^2\theta$$

$$\frac{dP}{d\Omega} = \frac{e^2 a^2\sin^2\theta}{4\pi c^3}$$ while the charge is accelerating

b. $$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} dt\,\frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt\,|\vec{A}(t)|^2$$

$$\vec{A}(t) = \left(\frac{c}{4\pi}\right)^{1/2}[R\vec{E}]_{ret}$$

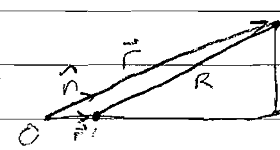
$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} d\omega\,|\vec{A}(\omega)|^2$ (Parseval's Theorem), where

$$\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\,\vec{A}(t)e^{i\omega t}$$

Then $\frac{dE}{d\Omega} = \int_0^{\infty} d\omega\,\frac{d^2I}{d\omega\,d\Omega}$, $\quad \frac{d^2I}{d\omega\,d\Omega} = 2|\vec{A}(\omega)|^2$, energy per unit solid angle per frequency interval

$$\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{c}{4\pi}\right)^{1/2}\cdot\int_{-\infty}^{\infty} dt\,e^{i\omega t}\cdot\frac{e}{c}\cdot\left(\hat{n}\times(\hat{n}\times\frac{\vec{a}}{c})\right)_{ret}$$

$$= \left(\frac{e^2}{8\pi^2 c^3}\right)^{1/2}\int_{-\infty}^{\infty} dt\,e^{i\omega t}\left(\hat{n}\times(\hat{n}\times\vec{a})\right)_{ret}$$



let $t = t' + \frac{R(t')}{c}$ retarded time: change of variables

$R(t') \approx r - \hat{n}\cdot\vec{r}\,'(t')$ $\quad \frac{dR}{dt'} \approx -\hat{n}\cdot\vec{v}\,'(t') \Rightarrow dt = dt'\left(1-\hat{n}\cdot\frac{\vec{v}\,'}{c}\right) \simeq dt'$

electric field only exists from 0 to Δt in retarded time

$$\vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c^3}\right)^{1/2} \int_0^{\Delta t} dt' \quad e^{i\omega t'} e^{-i\omega \frac{\hat{n}\cdot\vec{r}'(t')}{c}} \left(\hat{n}\times(\hat{n}\times\vec{a})\right)$$

ignoring overall phase factor $e^{i\omega r}$

$$\vec{r}'(t') = \tfrac{1}{2}\vec{a}\,t'^2 \qquad t' - \frac{\frac{1}{2}at'^2}{c} \approx t', \quad \text{since } \Delta t \text{ is tiny}$$

$$\vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c^3}\right)^{1/2} \cdot \hat{n}\times(\hat{n}\times\vec{a}) \cdot \frac{1}{i\omega}\left[e^{i\omega\Delta t} - 1\right]$$

$$|\vec{A}(\omega)|^2 = \frac{e^2}{8\pi^2 c^3} a^2 \sin^2\theta \cdot \frac{1}{\omega^2} \cdot \underbrace{\left[1 - e^{i\omega\Delta t} - e^{-i\omega\Delta t} + 1\right]}_{2 - 2\cos\omega\Delta t = 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\frac{\omega\Delta t}{2}\right) = 4\sin^2\frac{\omega\Delta t}{2}}$$

$$|\vec{A}(\omega)|^2 = \frac{e^2}{2\pi^2 c^3} a^2 \sin^2\theta \cdot \frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\omega^2}$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{\pi^2 c^3} a^2 \sin^2\theta \, \frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\omega^2}$$

integrate over solid angles; $\int \sin^2\theta\, d\Omega = \frac{8\pi}{3}$

$$E = \int_0^\infty \frac{dI}{d\omega} d\omega, \qquad \frac{dI}{d\omega} = \frac{8e^2 a^2}{3\pi c^3} \frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\omega^2}$$

Change variables to wavelength λ: $\omega = \frac{2\pi c}{\lambda} \qquad d\omega = -\frac{2\pi c\, d\lambda}{\lambda^2}$

$$E = \int_0^\infty \frac{8e^2 a^2}{3\pi c^3} \frac{\lambda^2}{(2\pi c)^2} \cdot \frac{2\pi c}{\lambda^2} \cdot \sin^2\left(\frac{\pi c \Delta t}{\lambda}\right) d\lambda$$

$$\frac{dI}{d\lambda} = \frac{4e^2 a^2}{3\pi^2 c^4} \sin^2\left(\frac{\pi c\Delta t}{\lambda}\right)$$

$V_f = a\Delta t$