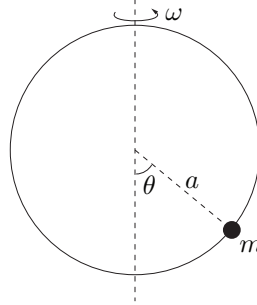


M08M.1 - Bead on a Hoop

Problem

A bead of mass m slides without friction on a circular loop of radius a and mass M . The loop lies in a vertical plane and rotates about a vertical diameter with angular velocity ω .

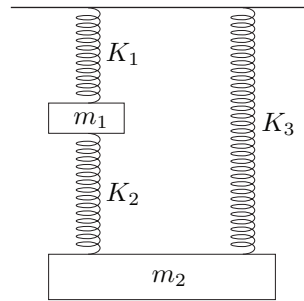


- Initially, prepare the system in a state such that $\omega = \text{constant}$ and the bead is at some stable equilibrium point $\theta_0(\omega)$. Find $\theta_0(\omega)$.
- Now assume $\omega^2 > g/a$, and move the bead from θ_0 by a small amount $\Delta\theta$. Suppose the bead then undergoes small oscillations around θ_0 . Find the condition under which we can treat the angular velocity approximately as constant.
- Find the frequency of small oscillations.

M08M.2 - Springs and Blocks

Problem

A coupled spring-block system hanging from the ceiling is shown in the figure below. The blocks are constrained to undergo only one dimensional vertical movement without rotation.



Suppose the system is oscillating around its equilibrium configuration. Find the normal modes.

M08M.3 - Gravitational Capture of Dark Matter Particles

Problem

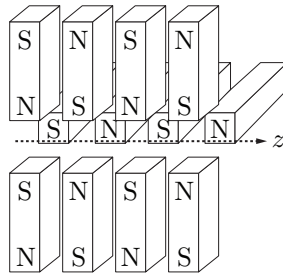
- a) A particle of mass m is in a circular orbit around the sun at the radius of the Earth's orbit, traveling at a speed $v_{\oplus} = 29.8$ km/s. Neglecting the Earth's gravitational potential, compute the escape speed v_{esc}^{\odot} from the solar system at the radius of the orbit. Compute the escape velocity in the frame of the Sun.
- b) What is the minimum speed v_{min} of a particle just escaping the solar system computed in a frame corotating with the particle's original circular orbit. Here, the corotating frame is the frame moving with the orbital velocity of the Earth. Neglect Earth's gravitational potential.
- c) Now assume that the particle is at rest in the bottom of the potential well of the Earth. If the escape speed from the Earth at the Earth's surface is $v_{\text{esc}}^{\oplus}(R_{\oplus}) = 11.2$ km/s, what is the escape speed $v_{\text{esc}}^{\oplus}(0)$ from the Earth for a particle at its center, computed in the Earth's frame? Assume the Earth is a sphere of uniform density. Here the Earth's radius is denoted R_{\oplus} .
- d) A particle of mass m has a velocity v relative to the Earth as it traverses the solar system at the orbital radius of the Earth around the Sun. The initial velocity v is the value far enough outside the gravitational well of Earth that the Earth's gravitational effects need to be accounted for in what follows.

The particle takes a trajectory that passes through the center of the Earth, and at the very center elastically scatters off of an iron atom, having a mass $m_{\text{Fe}} = 52 \text{ GeV}/c^2$. What is the maximum particle mass m that a particle can have and still be able to be gravitationally captured by the Earth subsequent to elastically scattering off of an iron atom at the center of the Earth? To compute m , assume the particle has the minimum velocity v such that it is not bound to the solar system.

M08E.1 - Wiggler

Problem

A wiggler magnet is constructed of alternating N-S dipole magnets.



An electron beam traveling in vacuum through the magnet in the z -direction and exactly on-axis ($x = 0, y = 0$) is “wiggled” by a magnetic field having components:

$$B_x(0, 0, z) = B_0 \cos kz ,$$

$$B_y(0, 0, z) = B_0 \sin kz ,$$

$$B_z(0, 0, z) = 0 .$$

Compute the off-axis B -field components, $B_x(x, y, z), B_y(x, y, z), B_z(x, y, z)$, within the vacuum region.

M08E.2 - Accelerating Point Charge

Problem

A particle with charge e starting at rest is given uniform acceleration, a , for a time Δt to non-relativistic energies.

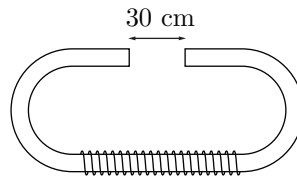
- a) Compute the power radiated per unit solid angle by the electric charge as a function of the angle θ measured with respect to its direction of acceleration.
- b) Assume the time Δt is infinitesimally short. Compute the total energy radiated per unit wavelength as a function of the final velocity v and wavelength λ of the radiated electromagnetic waves.

M08E.3 - Electromagnet with an Iron Core

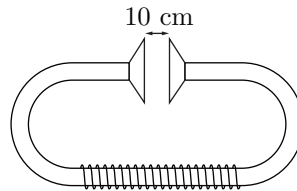
Problem

A long solenoid is made from $N = 1000$ turns of wire, wound at 10 turns per cm. Recall that $\mu_0 = 4\pi \cdot 10^{-7}$ T m/A.

- Give the approximate value in Tesla of the magnitude of the B -field at the center of the solenoid for $I = 100$ A of current.
- Insert a soft-iron core through the solenoid and bend the two ends together leaving a uniform gap distance of 30 cm. The total length L of the core is 3 meters and it has a constant cross-sectional area $A_{\text{core}} = 400$ cm². The relative permeability of the soft-iron is $\mu_r = 400$. Compute the magnitude of the B -field in the core B_{core} , and in the gap, B_{gap} , for a current $I = 100$ A. Assume that the B -field is uniform in the gap and there is no hysteresis.



- Assume the maximum value of $B_{\text{core}} = 1.5$ T. New pole-faces are added to the gap that shorten the gap to 10 cm, but increase the cross-sectional area at the gap to $A_{\text{pole}} = 1600$ cm². The cross-sectional area of the core is unchanged. Compute the maximum value of B_{gap} given the constraint on B_{core} and the new pole-face geometry. Assume that the B -field is uniform in the gap and there is no hysteresis.



M08Q.1 - Delta Function Potential

Problem

A particle of mass m is confined to a one dimensional space with potential

$$V(x) = -V_0[\delta(x + a) + \delta(x - a)].$$

- a) Write the general form of a bound state solution. Find the boundary condition at $x = \pm a$.

Based on the symmetry of the problem, the solutions can be classified by their properties under the parity transformation $x \rightarrow -x$.

- b) For even parity solutions, $\psi(-x) = \psi(x)$, show that there is always one (and only one) bound state solution.
- c) For odd parity solutions, $\psi(-x) = -\psi(x)$, determine the condition under which there is a bound state solution.

M08Q.2 - Ladder Operators

Problem

The dynamics of a system is characterized by the Hamiltonian

$$H = a^\dagger a + \frac{1}{2}, \quad [a, a^\dagger] = 1.$$

- a) Show that the ground state of this system, $|0\rangle$, satisfies

$$a|0\rangle = 0.$$

- b) Consider the state

$$|\alpha\rangle = \mathcal{N} e^{\alpha a^\dagger - \alpha^* a} |0\rangle,$$

where \mathcal{N} is some normalization constant. Show that $a|\alpha\rangle = \alpha|\alpha\rangle$. Find \mathcal{N} .

- c) Consider the change of variables

$$a = \frac{1}{\sqrt{2}}(q + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip).$$

Derive and interpret the hamiltonian in this set of new variables.

- d) Calculate $\langle \alpha|q|\alpha\rangle$. Describe the time dependence of $\langle \alpha|q|\alpha\rangle$.

M08Q.3 - Dynamics of Spin

Problem

A spin of $s = 1/2$ has its z -component “up” at time $t = 0$. The dynamics of the spin are given by the Hamiltonian

$$H = \lambda \hbar \sigma_x,$$

where σ_x is the usual Pauli matrix for a spin-1/2.

- a) If the z -component of this spin is measured at time $t = \tau$, what are the probabilities of each possible result of this measurement?
- b) Now consider a slightly different question: the spin again starts “up” at time $t = 0$, but now its z -component is measured twice, once at time $t = \tau/2$ and then again at time $t = \tau$. The above Hamiltonian gives the spin’s dynamics between the measurements, and you can assume the measurements happen instantaneously. However, the result of the first measurement at time $t = \tau/2$ is not known to you. Now what are the probabilities of each possible result of the second measurement at time $t = \tau$?

M08T.1 - Brownian Motion

Problem

A spherical object of radius a and mass M is undergoing Brownian motion at temperature T in a fluid of viscosity η . The fluid and the object have identical density, so there is no buoyancy force. When a is large compared to the size of the molecules in the fluid, this motion is given by the Langevin equation of motion for the velocity $\vec{v}(t)$ of the object:

$$M \frac{d\vec{v}(t)}{dt} = \vec{F}(t) - 6\pi\eta a \vec{v}(t),$$

where $\vec{F}(t)$ is the instantaneously random force on the object at time t due to collisions with the molecules of the fluid.

- a) Solving this Langevin equation (and also using equipartition of energy), calculate the autocorrelation function $\langle \vec{v}(0) \cdot \vec{v}(t) \rangle$ of the object's velocity.
- b) Use your result from (a) to calculate the leading long-time behavior of the mean-square displacement of this object after time t , namely $\langle (\vec{r}(t) - \vec{r}(0))^2 \rangle$, and show how measuring this quantity permits an experimental determination of Boltzmann's constant k_B .

M08T.2 - van der Waals Gas

Problem

The van der Waals equation of state is

$$P = \frac{Nk_B T}{V - Nb} - a \frac{N^2}{V^2}$$

for the pressure P of a fluid N interacting atoms in a volume V at temperature T . This models the liquid-gas phase transition and its critical point.

- a) **Briefly** explain the physics of each of the two above corrections to the ideal gas equation of state (corresponding to the parameters b and a).
- b) Calculate the parameters at the critical point: the critical pressure P_c , critical temperature T_c , and the critical density $n_c = (N/V)_c$.

M08T.3 - Electrons Escaping Metal

Problem

Assume that, to escape from a metal, an electron must impinge from the interior onto the surface with enough momentum to overcome the confining potential that holds the electrons in the metal. Also assume that all electrons with such a momentum do escape. Calculate the flux (number per area per time) of electrons escaping from a metal with work function ϕ (the barrier energy) at room temperature T . Treat the electrons as an ideal Fermi gas.