1 May 2007, Thermodynamics, Problem 1

1.1 (a)

\[ U = \int_{0}^{\infty} g(\omega) \bar{n}_{BE}(\omega) \omega \, d\omega = \int_{0}^{\infty} \frac{3V\omega^2}{2\pi^2} \frac{\omega}{e^{\omega/T} - 1} \, d\omega = \frac{VT^4\pi^2}{10} \]

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V = \frac{2VT^3\pi^2}{5} \]

\[ S = \int_{0}^{T} C_V(T') \frac{dT'}{T'} = \frac{2V\pi^2T^3}{15} \]

\[ F = U - TS = -\frac{VT^4\pi^2}{30} \]

\[ P = -\left( \frac{\partial F}{\partial V} \right)_T = \frac{T^4\pi^2}{30} \]  

(1)

1.2 (b)

Now we have two sub-systems:

\[ U = U_{bosons} + U_{fermions} \]

\[ U_f = \int_{0}^{\infty} \frac{24V\omega^2}{2\pi^2} \frac{\omega}{e^{\omega/T} + 1} \, d\omega = \frac{V(kT)^4\pi^2}{10} \]

\[ U_b = \int_{0}^{\infty} \frac{16V\omega^2}{2\pi^2} \frac{\omega}{e^{\omega/T} - 1} \, d\omega = \frac{8V(kT)^4\pi^2}{15} \]

\[ U = \frac{\pi^2VT^437}{30} \]

\[ C_V = \frac{2\pi^2Vk^4T^337}{15} \]

\[ S = \frac{74\pi^2Vk^4T^3}{45} \]

\[ F = -\frac{37\pi^2VT^4}{90} \]

\[ P_{\text{plasma}} = \frac{37\pi^2T^4}{90} \]  

(2)
1.3  (c)

\[ F = BV - \frac{37\pi^2VT^4}{90} \]

\[ P_{\text{plasma}} = \frac{37\pi^2T^4}{90} - B \]  \hspace{1cm} (3)

1.4  (d)

The meson phase will be stable when its free energy is lower than the free energy of the plasma phase:

\[ F_\pi \leq F_p \]

which implies:

\[ \frac{17\pi^2T^4}{45} \leq B \quad \text{Stable meson phase} \]  \hspace{1cm} (4)

At higher temperatures, when the relation is not satisfied any longer, the plasma phase will be stable.