1 May 2007, Quantum Mechanics, Problem 2

1.1 (a)

Use:

$$\frac{d\sigma}{d\Omega} = |f|^2$$
$$f = -\frac{m}{2\pi}\int V(\boldsymbol{r}')e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}'}\,d^3r'$$
$$\boldsymbol{k}' = p_0\hat{z}$$
$$\boldsymbol{k} = p_0\hat{r}$$

Let $\hat{z}'$ be the direction of the vector $\boldsymbol{k}'-\boldsymbol{k}$. Then we can define primed spherical coordinates with respect to this $\hat{z}'$ and we can use those to integrate. Furthermore:

$$\boldsymbol{k}'-\boldsymbol{k} = 2p_0 sin(\theta/2)\hat{z}$$
$$f = mV_0\int_0^R\int_0^\pi e^{i2p_0 sin(\theta/2)r'cos\theta'}r'^2 sin\theta'\,d\theta'\,dr'$$
$$f = \frac{2mV_0R^3}{[2p_0Rsin(\theta/2)]^3}\left[sin[2p_0sin(\theta/2)R] - 2p_0sin(\theta/2)Rcos[2p_0sin(\theta/2)R]\right]$$

In order to use the approximation $p_0R << 1$, we need to expand the sine and the cosine to fifth and fourth order, respectively:

$$\sin(x) - x\cos(x) \approx \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) - x\left(1 - \frac{x^2}{2} + \frac{x^4}{4!}\right) = \frac{x^3}{3}\left(1 - \frac{x^2}{10}\right)$$
$$f = \frac{2mV_0R^3}{3}\left(1 - \frac{[2p_0Rsin(\theta/2)]^2}{10}\right)$$
$$\frac{d\sigma}{d\Omega} = \frac{4m^2V_0^2R^6}{9}\left(1 - \frac{[2p_0Rsin(\theta/2)]^2}{5}\right) \tag{1}$$

1.2 (b)

There is a uniform background in the differential cross-section, with an angular dependence on top of that. This angular dependence goes like $-sin^2(\theta/2)$, so that means that backwards scattering is suppressed and forwards has the highest probability, with a monotonic transition in between these two points. In other words, the graph of $\frac{d\sigma}{d\Omega}$ versus θ between 0 and 2π looks like a very wide U.

1.3 (c)

Each potential well suppresses backwards scattering a little bit more, so in the end backwards scattering is totally suppressed, while forwards scattering is very high. Furthermore, there will be a diffraction pattern superimposed with this suppression shape.