May 2007 #1 (QM)

a. $F(\vec{r},\vec{p})$

$\frac{\partial F}{\partial t}=0$ $\quad \frac{d}{dt}\left(\langle \psi_n|F|\psi_n\rangle\right) = \langle \frac{d\psi_n}{dt}|F|\psi_n\rangle + \langle \psi_n|F|\frac{d\psi_n}{dt}\rangle$

$i\hbar \frac{d|\psi_n\rangle}{dt} = H|\psi\rangle \qquad |\frac{d\psi_n}{dt}\rangle = \frac{-i}{\hbar} H|\psi_n\rangle$

$= \langle \frac{-i}{\hbar} H\psi_n|F|\psi_n\rangle + \langle \psi_n|F|\frac{-i}{\hbar}H\psi_n\rangle$

$= \frac{i}{\hbar} E_n \langle \psi_n|F|\psi_n\rangle - \frac{i}{\hbar} E_n \langle \psi_n|F|\psi_n\rangle = 0$

$\frac{d\langle F\rangle}{dt} = 0$

b. $H = \frac{p^2}{2m} + V(\vec{r})$

let $F = \vec{r}\cdot\vec{p} = x_j p_j$ For any operator and any state,

$$\frac{d}{dt}\langle F\rangle = \frac{i}{\hbar}\langle [H,F]\rangle + \langle \frac{\partial F}{\partial t}\rangle$$

Let us compute this for an eigenstate, where we know $\frac{d\langle F\rangle}{dt} = 0$

$0 = \frac{i}{\hbar}\langle [H, x_j p_j]\rangle \qquad [H, x_j p_j] = \frac{1}{2m}[p^2, x_j p_j] + [V, x_j p_j]$

$[p^2, x_j p_j] = p^2 x_j p_j - x_j p_j p^2 = -\hbar^2\cdot(-i\hbar)\left[\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} x_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_j}\frac{\partial^2}{\partial x_k \partial x_k}\right] f$

$= -i\hbar\cdot -\hbar^2 \cdot \left[\underbrace{\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left(x_j \frac{\partial f}{\partial x_j}\right)}_{} - x_j \frac{\partial}{\partial x_j}\frac{\partial^2 f}{\partial x_k \partial x_k}\right]$

$\frac{\partial}{\partial x_k}\left\{\delta_{jk}\frac{\partial f}{\partial x_j} + x_j \frac{\partial^2 f}{\partial x_k \partial x_j}\right\} = \frac{\partial^2 f}{\partial x_k \partial x_k} + \frac{\partial^2 f}{\partial x_k \partial x_k} + x_j \frac{\partial}{\partial x_k}\frac{\partial^2 f}{\partial x_k \partial x_j}$

$= i\hbar\cdot(-\hbar^2)\cdot 2\nabla^2 f = -2i\hbar\cdot(-\hbar^2\nabla^2) f = -2i\hbar\, p^2 f$

$\Rightarrow [p^2, x_j p_j] = -2i\hbar p^2$

$[V, x_j p_j] = V x_j p_j - x_j p_j V = x_j V p_j - x_j p_j V = x_j [V, p_j]$ since $[x_j, V] = 0$

$[V, p_j] f = -i\hbar\left[V\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} V\right] f = -i\hbar\left[V\frac{\partial f}{\partial x_j} - V\frac{\partial f}{\partial x_j} - f\frac{\partial V}{\partial x_j}\right] = i\hbar \frac{\partial V}{\partial x_j} f$

$[V, p_j] = i\hbar \frac{\partial V}{\partial x_j}$

$x_j [V, p_j] = i\hbar x_j \frac{\partial V}{\partial x_j} = i\hbar\, \vec{r}\cdot\nabla V$

$$0 = \langle -2i\hbar \frac{p^2}{2m} \rangle + \langle i\hbar\, \vec{r}\cdot\nabla V \rangle$$

$$\Rightarrow 2\langle \frac{p^2}{2m} \rangle = \langle \vec{r}\cdot\nabla V \rangle$$ for eigenstates $|\psi_n\rangle$

c. constant force, $V = -Fr$

$-\nabla V = F\hat{r}$

$\vec{r}\cdot\nabla V = rF = V$

$\langle T \rangle = \frac{1}{2}\langle \vec{r}\cdot\nabla V \rangle = \frac{1}{2}\langle V \rangle$

$m_0 = E \sim \langle T \rangle + \frac{1}{2}\langle V \rangle = \frac{3}{2}\langle V \rangle$

$\langle V \rangle = \frac{2}{3} m_\sigma$