1 May 2007, Quantum Mechanics, Problem 1

1.1 (a)

Heisenberg's equation reads:

$$\frac{d}{dt} < F >= -i < [F, H] > + \left(\frac{\partial F}{\partial t}\right)_{classical}$$

In our case, the last term is 0, and because we are working with eigenstates:

$$< [F, H] >=< \psi_n|FH|\psi_n > - < \psi_n|HF|\psi_n >=< \psi_n|F|\psi_n > E_n - E_n < \psi_n|F|\psi_n >= 0$$

1.2 (b)

Note that the result $< \psi_n|[F, H]|\psi_n >= 0$ holds for any operator F, and so $< \psi_n|[\boldsymbol{r} \cdot \boldsymbol{p}, H]|\psi_n >= 0$. At the same time:

$$< \psi_n|[\boldsymbol{r} \cdot \boldsymbol{p}, H]|\psi_n >=< \psi_n|[xp_x, H]|\psi_n > + < \psi_n|[yp_y, H]|\psi_n > + < \psi_n|[zp_z, H]|\psi_n >$$

$$< [xp_x, H] >=< x[p_x, H] + [x, H]p_x >$$

$$[p_x, H] = -i\frac{\partial V}{\partial x}$$

$$[x, H] = \frac{ip_x}{m}$$

$$\left\langle x(-i)\frac{\partial V}{\partial x} + \frac{ip_x}{m}p \right\rangle = 0$$

$$\left\langle \frac{p_x^2}{m} \right\rangle = \left\langle x\frac{\partial V}{\partial x} \right\rangle \tag{1}$$

which naturally generalizes to the vector case.

1.3 (c)

Since the force is independent of distance, so the potential must be something like:

$$V = -\boldsymbol{r} \cdot \boldsymbol{F}$$

$$\boldsymbol{r} \cdot \nabla V = -\boldsymbol{r} \cdot \boldsymbol{F} = V$$

$$< T >= \frac{1}{2} < V >$$

At rest, all the energy of the proton must be internal:

$$E = < T > + < V > = \frac{3}{2} < V >$$

But also at rest we know that the energy of the proton is equal to its mass. Therefore:

$$< V > = \frac{2}{3} E = \frac{2}{3} m_P \qquad (2)$$