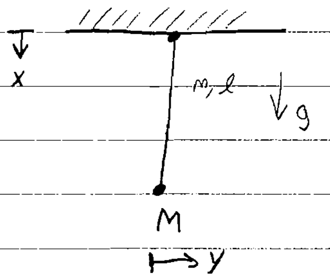
May 2007 #3 (CM)

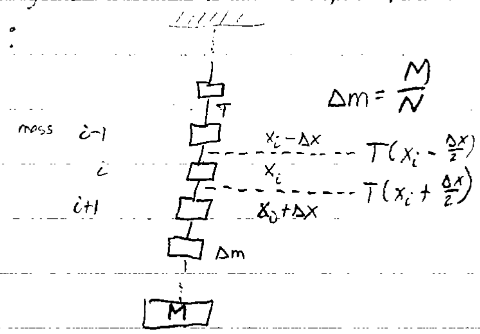
mass M hanging from a string of mass m, length l, gravity g

upper end of string fixed, $M \gg m$



a. Model the string as N elements Δm, separated by Δx, connected by strings with tension $T(x)$:

$$\Delta m = \frac{M}{N} \qquad \Delta x = \frac{l}{N} \qquad (\text{take limit as } N \to \infty)$$

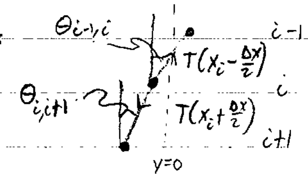


°Assume small transverse oscillations:

Force balance on mass i: $\Delta m \, \vec{a}_i = \vec{F}_i$

$F_{ix} = \Delta m g + T_{ix}$

$F_{iy} = T_{iy}$



Let y_i be the horizontal displacement of mass i

$$T_{ix} = -T\left(x_i - \frac{\Delta x}{2}\right)\cos\theta_{i-1,i} + T\left(x_i + \frac{\Delta x}{2}\right)\cos\theta_{i,i+1}$$

$$T_{iy} = T\left(x_i - \frac{\Delta x}{2}\right)\sin\theta_{i-1,i} - T\left(x_i + \frac{\Delta x}{2}\right)\sin\theta_{i,i+1}$$

$$\sin\theta_{i-1,i} = \frac{y_{i-1} - y_i}{\sqrt{\Delta x^2 + (y_{i-1} - y_i)^2}} \qquad \cos\theta_{i-1,i} = \frac{\Delta x}{\sqrt{\Delta x^2 + (y_{i-1} - y_i)^2}}$$

Under the small transverse oscillation assumption, the angles are small, and to first order, $\sin\theta_{i-1,i} = \dfrac{y_{i-1} - y_i}{\Delta x}$, $\cos\theta_{i-1,i} = 1$

Taylor Expanding $T\left(x_i - \frac{\Delta x}{2}\right)$ about x_i,

$$T_{ix} = T(x_i) + \frac{\partial T}{\partial x}\frac{\Delta x}{2} - T(x_i) + \frac{\partial T}{\partial x}\frac{\Delta x}{2} = \frac{\partial T}{\partial x}\Delta x$$

$$T_{iy} = \left(T - \frac{\partial T}{\partial x}\frac{\Delta x}{2}\right)\frac{y_{i-1} - y_i}{\Delta x} - \left(T + \frac{\partial T}{\partial x}\frac{\Delta x}{2}\right)\frac{y_i - y_{i+1}}{\Delta x} = T\,\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x} + \frac{\partial T}{\partial x}\,\frac{y_{i+1} - y_{i-1}}{2}$$

$$T_{ix} = \frac{dT}{dx}\Delta x$$

$$T_{iy} = \Delta x\, T \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} + \Delta x \frac{dT}{dx} \frac{y_{i+1} - y_{i-1}}{2\Delta x}$$

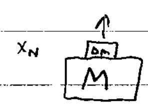
$$\Delta m\, \ddot{x}_i = \Delta m\, g + \Delta x \frac{dT}{dx}$$

In small oscillations, the vertical displacement is negligible, so $\ddot{x}_i = 0$

$$0 = \frac{m}{N} g + \frac{\ell}{N} \frac{dT}{dx}$$ [Passage to continuum limit, becomes exact]

$$\frac{dT}{dx} = \frac{-mg}{\ell} = -\mu g$$ where $\mu = \frac{m}{\ell}$ is mass per unit length

Boundary conditions: look at the last mass $\Delta m + M$ at the bottom



$\uparrow T(\ell)$

$\downarrow Mg + \Delta mg$

$$T(\ell) = Mg + \Delta mg \cong Mg$$

$$T = -\frac{mgx}{\ell} + C \qquad T(\ell) = Mg = -mg + C \qquad C = Mg + mg$$

$$T(x) = Mg + mg\left(1 - \frac{x}{\ell}\right) = g\left[M + m\left(1 - \frac{x}{\ell}\right)\right]$$

$$T(x) \approx Mg \quad \text{for } M \gg m \text{ (constant)}$$

$$\Delta m\, \ddot{y}_i = \Delta x\, T \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} + \Delta x \frac{dT}{dx} \frac{y_{i+1} - y_{i-1}}{2\Delta x}$$

$N \to \infty$, continuum

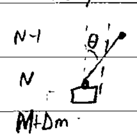
$y_i(t) \to y(x,t)$

$$\frac{m}{N}\ddot{y} = \frac{\ell}{N} T \frac{\partial^2 y}{\partial x^2} + \frac{\ell}{N} \frac{dT}{dx} \frac{\partial y}{\partial x}$$

T is approx. constant, so $\frac{dT}{dx} \sim$ negligible [that term can be shown to be smaller by $\frac{m}{M}$)

$$\frac{m}{\ell T} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \qquad \boxed{\frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}} \qquad v^2 = \frac{\ell T}{m} = \frac{T}{\mu} = \frac{\ell g M}{m}$$

Boundary conditions: $y(0,t)$ (fixed) $\qquad y(\ell,t)$:



$$(M + \Delta m)\ddot{y}_N = T \frac{y_{N-1} - y_N}{\Delta x} \to M \frac{\partial^2 y}{\partial t^2} = -T \frac{\partial y}{\partial x}(\ell, t)$$

$$\Rightarrow \frac{\partial^2 y(\ell,t)}{\partial t^2} = -g \frac{\partial y(\ell,t)}{\partial x}$$

b. $\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$ normal modes are oscillations' sines/cosines

$y(0)=0$, so use $\sin kx$

$y(x,t) = \sin(kx)\cos(\omega t+\phi)$ $\quad \omega^2 = k^2 v^2$

$$\frac{\partial^2 y}{\partial t^2}(\ell,t) = -g\frac{\partial y}{\partial x}(\ell,t)$$

$$-\omega^2 \sin(k\ell) = gk\cos k\ell$$

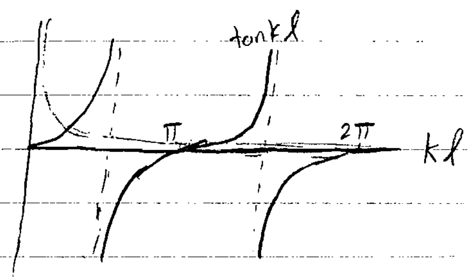
$$-k^2v^2\sin(k\ell) = gk\cos k\ell$$

$$\Rightarrow \tan k\ell = +\frac{g}{kv^2} = +\frac{g\ell/v^2}{k\ell}$$

$$\frac{g\ell}{v^2} = \frac{g\ell}{\ell g M/m} = \frac{m}{M}$$

$$\boxed{\tan k\ell = \frac{+m/M}{k\ell}}$$ Equation giving the solutions for k

c. $\frac{m}{M}$ small:



Solve $\tan x = \frac{\alpha}{x}$ where $\alpha \ll 1$

The solutions are near $n\pi$, as can be seen graphically. For large n, the approximation $n\pi$ gets more accurate.

Linearize $\tan x$ about $n\pi$: $\tan(n\pi)=0$ $\quad \frac{d\tan(x)}{dx}\Big|_{n\pi} = \sec^2(n\pi) = 1$

$\tan x \approx (x-n\pi)$

$x - n\pi = \sim\frac{\alpha}{x}$ $\qquad x \simeq n\pi$, so replace $\frac{1}{x}$ with $\frac{1}{n\pi}$

$x = n\pi + \frac{\alpha}{n\pi}$

First solution: $x^2 = \alpha = k^2\ell^2$

$$\omega = kv = \frac{\sqrt{\alpha}}{\ell}\cdot v = \frac{\sqrt{\frac{m}{M}}}{\ell}\cdot\sqrt{\frac{\ell g M}{m}} = \sqrt{\frac{g}{\ell}}$$

Second solution: $x = \pi + \frac{\alpha}{\pi}$ $\qquad k\ell \simeq \pi\left(1+\frac{\alpha}{\pi^2}\right)$ $\qquad \frac{2\pi}{\lambda} = \frac{\pi}{\ell}\left(1+\frac{\alpha}{\pi^2}\right)$

$$\lambda = \frac{2\ell}{\left(1+\frac{\alpha}{\pi^2}\right)}$$ Wavelength nearly 2ℓ.