1 May 2007, Mechanics, Problem 3

1.1 (a)

Define x as the coordinate that goes downward and y as the coordinate that goes to the left. Take a small piece of string and write its horizontal and vertical equations of motion:

$$\tau cos\theta(x) = \tau cos(\theta(x + dx)) + \frac{m}{l} dxg$$

$$\tau sin(\theta(x))\theta'(x) = \frac{m}{l} g \quad Vertical$$

$$\frac{m}{l} dx\ddot{y} = \tau sin[\theta(x) + \theta'(x)dx] - \tau sin\theta(x) = \tau cos\theta(x)\theta'(x)dx$$

$$\frac{m\ddot{y}}{l} = \tau cos\theta\theta' \quad Horizontal$$

In all these equations, θ is defined as the angle that a differential piece of string makes with the vertical line. We can express it in terms of y as:

$$tan\theta = y'$$

$$sec^2\theta\theta' = y''$$

Plug this in to get:

$$\tau sin\theta y'' cos^2\theta = \frac{mg}{l}$$

$$\frac{m\ddot{y}}{l} = \tau y'' cos^3\theta$$

At the top we have y=0, and at the bottom:

$$\tau cos\theta(l) = Mg$$

$$-\tau sin\theta(l) = M\ddot{y}(l)$$

For small oscillations, these become:

$$\frac{m\ddot{y}}{l} = \tau y'' \quad (1)$$

$$\tau = Mg \quad (2)$$

$$-\tau y'(l) = M\ddot{y}(l) \quad (3)$$

$$y(0) = 0 \quad (4)$$

1.2 (b)

Let $v^2 \equiv \frac{\tau l}{m}$, and let $k \equiv \omega/v$. Then a wavemode with frequency ω has a form:

$$y(x,t) = Asin(\omega t + \phi)sin(kx + \varphi)$$
$$y(0,t) = Asin(\omega t + \phi)sin\varphi = 0 \rightarrow \varphi = 0 \quad (4)$$
$$kgcos(kl) = \omega^2 sin(kl) \rightarrow (kl)tan(kl) = \frac{m}{M} \quad (3)$$
$$(\omega l/v)tan(\omega l/v) = \frac{m}{M} \tag{5}$$

1.3 (c)

To lowest order we get $\omega = 0$, but then there is no motion. To first order in m/M, we get:

$$(kl)^2 = \frac{m}{M}$$
$$\omega = \sqrt{\frac{g}{l}} \tag{6}$$

The whole system swings like a pendulum of length l. The next lowest frequency will be for the case when (kl) is not small but tan(kl) is:

$$kl \approx \pi + \epsilon$$
$$\omega = \pi\sqrt{\frac{Mg}{ml}} \tag{7}$$

In this case, the point mass remains fixed, acting as a node, while the string oscillates back and forth. You can see that the frequency is much larger than the one for pendulum motion.