1 May 2007, Mechanics, Problem 3

1.1 (a)

Define \( x \) as the coordinate that goes downward and \( y \) as the coordinate that goes to the left. Take a small piece of string and write its horizontal and vertical equations of motion:

\[
\tau \cos \theta(x) = \tau \cos(\theta(x + dx)) + \frac{m}{l} dx g \\
\tau \sin (\theta(x)) \theta'(x) = \frac{m}{l} \, g 
\]

Vertical

\[
\frac{m}{l} \, dy = \tau \sin[\theta(x) + \theta'(x)dx] - \tau \sin \theta(x) = \tau \cos \theta(x) \theta'(x)dx
\]

Horizontal

\[
\frac{m y}{l} = \tau \cos \theta' 
\]

In all these equations, \( \theta \) is defined as the angle that a differential piece of string makes with the vertical line. We can express it in terms of \( y \) as:

\[
tan \theta = y' \\
sec^2 \theta' = y''
\]

Plug this in to get:

\[
\tau \sin \theta y'' \cos^2 \theta = \frac{mg}{l} \\
\frac{m y}{l} = \tau y'' \cos^3 \theta
\]

At the top we have \( y=0 \), and at the bottom:

\[
\tau \cos \theta(l) = M g \\
-\tau \sin \theta(l) = M \ddot{y}(l)
\]

For small oscillations, these become:

\[
\frac{m \ddot{y}}{l} = \tau y'' \tag{1} \\
\tau = M g \tag{2} \\
-\tau y'(l) = M \ddot{y}(l) \tag{3} \\
y(0) = 0 \tag{4}
\]
1.2 (b)
Let $v^2 \equiv \frac{\tau}{m}$, and let $k \equiv \omega/v$. Then a wavemode with frequency $\omega$ has a form:

$$y(x,t) = A\sin(\omega t + \phi)\sin(kx + \varphi)$$

$$y(0,t) = A\sin(\omega t + \phi)\sin\varphi = 0 \rightarrow \varphi = 0 \quad (4)$$

$$kg\cos(kl) = \omega^2 \sin(kl) \rightarrow (kl)\tan(kl) = \frac{m}{M} \quad (3)$$

$$(\omega l/v)\tan(\omega l/v) = \frac{m}{M} \quad (5)$$

1.3 (c)
To lowest order we get $\omega = 0$, but then there is no motion. To first order in $m/M$, we get:

$$(kl)^2 = \frac{m}{M}$$

$$\omega = \sqrt{\frac{g}{l}} \quad (6)$$

The whole system swings like a pendulum of length $l$. The next lowest frequency will be for the case when $(kl)$ is not small but $\tan(kl)$ is:

$$kl \approx \pi + \epsilon$$

$$\omega = \pi \sqrt{\frac{Mg}{ml}} \quad (7)$$

In this case, the point mass remains fixed, acting as a node, while the string oscillates back and forth. You can see that the frequency is much larger than the one for pendulum motion.