1 May 2007, Mechanics, Problem 2

1.1 (a)

Since there is no gravity, the Lagrangian is simply:

$$\mathcal{L} = \frac{1}{2}m\dot{\boldsymbol{x}}^2$$

$$\dot{\boldsymbol{x}} = \left(acos\theta cos\phi\dot{\theta} - a\sin\theta sin\phi\dot{\phi}\,,\; acos\theta\dot{\theta}sin\phi + asin\theta cos\phi\dot{\phi}\,,\; -bsin\theta\dot{\theta}\right)$$

$$\dot{\boldsymbol{x}}^2 = a^2cos^2\theta\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2 + b^2sin^2\theta\dot{\theta}^2$$

$$\mathcal{L} = \frac{1}{2}m\left(a^2cos^2\theta\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2 + b^2sin^2\theta\dot{\theta}^2\right) \tag{1}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$-a^2\dot{\theta}^2 + a^2cot\theta\ddot{\theta} + b^2\dot{\theta}^2 + b^2tan\theta\ddot{\theta} = a^2\dot{\phi}^2 \tag{2}$$

$$\frac{dA}{dt} = 0 \quad A = \dot{\phi}sin^2\theta \tag{3}$$

1.2 (b)

Notice that since there is no potential energy, the total energy is just the kinetic energy, and it equals the lagrangian:

$$E = \mathcal{L} = \frac{1}{2}m\left[a^2\left(cos^2\theta\dot{\theta}^2 + \frac{A^2}{sin^2\theta}\right) + b^2sin^2\theta\dot{\theta}^2\right]$$

The endpoints of the motion will be those for which the particle is instantaneously at rest, i.e., $\dot{\theta} = 0$. Those are found at:

$$E = \frac{1}{2}m\left[a^2\left(\frac{A^2}{sin^2\theta}\right)\right]$$

$$2Esin^2\theta = ma^2A^2$$

Then we can solve the energy equation for $\dot{\theta}$:

$$\dot{\theta}^2 = \frac{2Esin^2\theta - ma^2A^2}{msin^2\theta(a^2cos^2\theta + b^2sin^2\theta)} \equiv -V_{E,A}$$

The period of motion is:

$$T = \int_0^T dt = \oint \frac{dt}{d\theta} d\theta$$

We can visualize the motion as being from one endpoint to the other and then back, or equivalently twice between endpoints (since the energy is conserved, the integral is independent of path):

$$T = 2 \int_{\theta_-}^{\theta_+} \frac{d\theta}{\sqrt{-V_{E,A}}}$$

This completes the proof, after reminding that θ_- and θ_+ are the roots of the equation that defines the endpoints of the motion:

$$\theta_{\pm} = arcsin \left(\pm \sqrt{\frac{ma^2 A^2}{2E}} \right)$$

Note that these are also the roots of $V_{A,E} = 0$, so the proof is complete.