1 May 2007, Mechanics, Problem 1

1.1 (a)

We use one of Kepler's laws, that says:

$$T^2 = ka^3$$

where a is the semimajor axis of the ellipse and T is the period of the motion. To derive what the proportionality constant k is, we use the case of a circle:

$$ma\left(\frac{2\pi}{T}\right)^2 = m\omega^2 a = \frac{GMm}{a^2}$$
$$a^3\frac{4\pi^2}{gR_0^2} = T^2$$

where R_0 is the radius of the earth.

$$T_{transfer} = \frac{T}{2} = \frac{\pi}{R_0}\sqrt{\frac{(R_0+R_1)^3}{8g}} \tag{1}$$

1.2 (b)

This problem has no relation with the previous one, as far as I can tell. Use the assumption that the shower of asteroids is much larger than the earth, but the asteroids themselves are much smaller than the earth. Then we can view this as a point scattering problem. Let m be the mass of each asteroid, d be the distance of closest approach, let b be the impact parameter, and let v_f be the velocity at the point of closest approach. Then conservation of angular momentum and energy give:

$$mvb = mv_f d$$
$$\frac{mv^2}{2} = -\frac{GMm}{d} + \frac{mv_f^2}{2}$$

Solve the first equation for v_f and plug into the second equation. The largest possible impact parameter such that the asteroid will hit the earth will be for the case when the distance of closest approach is exactly the radius of the earth. So plug $d = R_0$ in the second equation to obtain:

$$b_{max} = \sqrt{\frac{(R_0v^2 + gR_0^2)^2 - (gR_0^2)^2}{v^4}}$$
$$\sigma = \pi b_{max}^2 = \pi R_0\frac{R_0v^2 + 2gR_0^2}{v^2} \quad Cross-section$$
$$N = n\sigma = n\pi R_0^2\left(1 + \frac{2gR_0}{v^2}\right) \tag{2}$$