1 May 2007, Electromagnetism, Problem 3

1.1 (a)

We need to solve Maxwell's equations inside the conductor:

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

It is the continuity of the permittivity and permeability that allows us to use these equations instead of their "inside matter" counterparts. Take the curl of the second equation and plug in the fourth one:

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t}\left(\mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right)$$
$$\nabla^2 \boldsymbol{E} = \mu_0 \sigma \frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

Plug in a solution of the form:

$$\boldsymbol{E} = \boldsymbol{E}_T e^{i(kx-\omega t)} e^{-mx}$$
$$\boldsymbol{E}_T (ik - m)^2 e^{i(kx-\omega t)} e^{-mx} = \mu_0 \sigma (-i\omega) \boldsymbol{E}_T e^{i(kx-\omega t)} e^{-mx} + \frac{1}{c^2}(-\omega^2) \boldsymbol{E}_T e^{i(kx-\omega t)} e^{-mx}$$
$$(k + im)^2 = k^2 - m^2 + 2ikm = i\mu_0 \sigma \omega + \frac{\omega^2}{c^2}$$

Some nasty algebra shows that:

$$m = \frac{\omega}{\sqrt{2}c}\sqrt{\sqrt{1 + \left(\frac{\mu_0 \sigma c^2}{\omega}\right)^2} - 1} \tag{1}$$
$$k = \frac{\omega}{\sqrt{2}c}\sqrt{\sqrt{1 + \left(\frac{\mu_0 \sigma c^2}{\omega}\right)^2} + 1} \tag{2}$$

Now, the incident wave was something like:

$$\boldsymbol{E}_{in} = \boldsymbol{E_0} e^{i(kx-\omega t)}$$

with $k = \omega/c$. The component of the electric field parallel to the surface must be continuous at the interface. Furthermore, the discontinuity in the perpendicular component of the displacement field is proportional to the free surface charge density, which is equal to 0 in our case. Therefore, we find that:

$$\boldsymbol{E}_T = \boldsymbol{E}_0$$
$$\boldsymbol{E}_{cond} = \boldsymbol{E}_0 e^{i(kx-\omega t)} e^{-mx} \tag{3}$$

with m and k given in equations (1) and (2). The magnetic field is obtained by exploiting the second Maxwell equation:

$$\boldsymbol{B} = -\int (\nabla \times \boldsymbol{E}) dt$$
$$\boldsymbol{B}_{cond} = (-E_{0_z}\hat{y} + E_{0_y}\hat{z})\frac{k + im}{\omega} e^{i(kx-\omega t)} e^{-mx} \tag{4}$$

For both fields, the actual solution is only the real part of what we obtained. Since $\boldsymbol{E}_0$ could in principle be complex, we won't write an explicit form for the real solution.

1.2 (b)

Let's suppose that the constants E_{0_y} and E_{0_z} are both real. Then:

$$\boldsymbol{E}_{cond} = \boldsymbol{E}_0 cos(kx - \omega t) e^{-mx}$$
$$\boldsymbol{B}_{cond} = (-E_{0_z}\hat{y} + E_{0_y}\hat{z})\frac{kcos(kx-\omega t) - msin(kx-\omega t)}{\omega} e^{-mx}$$

The Poynting vector is:

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} = [kcos^2(kx-\omega t) - mcos(kx-\omega t)sin(kx-\omega t)]e^{-2mx}\frac{E_0^2}{\omega\mu_0}\hat{x}$$

This represents energy flux per unit area per unit time. Let A be the cross-sectional area of the conductor. Then the energy lost per unit time in a stretch of length Δx is:

$$\Delta W = S(x)A - S(x+\Delta x)A = S(x)A - S(x)A - S'(x)\Delta x A = -S'(x)\Delta x A$$
$$\Delta W = -\Delta x A E_0^2 \frac{e^{-2mx}}{\omega\mu_0}[-2k^2 sin(kx-\omega t)cos(kx-\omega t) + msin^2(kx-\omega t) - mcos^2(kx-\omega t)$$
$$-2mkcos^2(kx-\omega t) + 2m^2 cos(kx-\omega t)sin(kx-\omega t)]$$

After averaging over one full cycle of time:

$$\Delta W = -\Delta x A E_0^2 \frac{e^{-2mx}}{\omega\mu_0}[m\frac{1}{2} - m\frac{1}{2} - 2mk\frac{1}{2}]$$
$$\Delta W = \Delta x A E_0^2 e^{-2mx}\frac{\sigma}{2} \tag{5}$$

The Ohmic power delivered is:

$$P = \int_{A,\Delta x} \boldsymbol{J}\cdot\boldsymbol{E}\, d\tau = \sigma\int_{A,\Delta x} \boldsymbol{E}^2\, d\tau = \sigma A\Delta x \boldsymbol{E}(x)^2 = \sigma A \Delta x E_0^2 e^{-2mx}\frac{1}{2} \tag{6}$$

Note the 1/2 factor after the last integral. That is due to the time averaging of the square of the electric field.