1 May 2007, Electromagnetism, Problem 1

1.1 (a)

This problem may seem confusing because the sphere is not hollow, but actually nothing changes with respect to the hollow case. The usual way to solve this problem is through the method of images. We place a charge $q' = -qR/d$ at a distance $a = R^2/d$ from the center of the sphere, along the line that joins the center of the sphere and the point charge outside the sphere. This generates a potential (outside the sphere):

$$V(r,\theta) = -\frac{q}{4\pi\epsilon_0\sqrt{R^2 + \frac{d^2r^2}{R^2} - 2rdcos\theta}} + \frac{q}{4\pi\epsilon_0\sqrt{d^2 + r^2 - 2drcos\theta}}$$

This ensures that the potential is 0 on the surface of the sphere. But this would cause the sphere to have a total charge $-qR/d$, when in fact we know the sphere to have no charge. We fix this problem by placing an extra charge at the center of the sphere, of value $q'' = qR/d$. Therefore the total potential outside of the sphere is:

$$V(r,\theta) = -\frac{q}{4\pi\epsilon_0\sqrt{R^2 + \frac{d^2r^2}{R^2} - 2rdcos\theta}} + \frac{q}{4\pi\epsilon_0\sqrt{d^2 + r^2 - 2drcos\theta}} + \frac{qR}{4\pi\epsilon_0 rd}$$

The electric field at the location of the point charge is given by the other two charges:

$$\boldsymbol{E}(d) = \left(\frac{qR/d}{4\pi\epsilon_0 d^2} - \frac{qR/d}{4\pi\epsilon_0(d - R^2/d)^2}\right)\hat{d} = \left(\frac{qR}{4\pi\epsilon_0 d^3} - \frac{qRd}{4\pi\epsilon_0(d^2 - R^2)^2}\right)\hat{d}$$

$$\boldsymbol{F}_{on\,charge} = \left(\frac{q^2R}{4\pi\epsilon_0 d^3} - \frac{q^2Rd}{4\pi\epsilon_0(d^2 - R^2)^2}\right)\hat{d} \tag{1}$$

1.2 (b)

We use the condition:

$$E^r_{out} - E^r_{in} = E^{||}_{out} - E^{||}_{in} = \frac{\sigma}{\epsilon_0}$$

$$E^r = -\frac{\partial V}{\partial r}$$

And we know that the electric field inside a conductor is $\mathbf{0}$, so taking V to be the total potential outside, as found in part a, in order to find the surface charge density, we need:

$$\sigma = -\epsilon_0\left(\frac{\partial V}{\partial r}\right)_{R,out}$$

$$\sigma(\theta) = \frac{q}{4\pi Rd} + \frac{q(R^2 - d^2)}{4\pi R(R^2 + d^2 - 2Rdcos\theta)^{3/2}} \tag{2}$$

And if we integrate this over the surface of the sphere, we check that indeed the total charge on the sphere is 0.