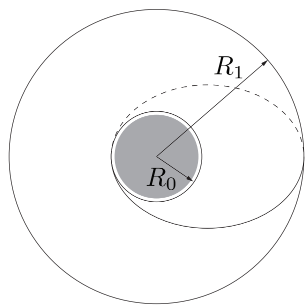
M07M.1 - Planetary Orbits

Problem

A satellite in a low Earth circular orbit with Radius R_0 has an orbital period T_0.

a) How long does it take to transfer the satellite into a new circular orbit with a larger radius R_1 using the Hohmann transfer ellipse shown in the figure?



b) Suppose a large shower of asteroids (much larger than the Earth diameter) came to Earth from a distant source, all moving with the same initial velocity v. If the areal number density of asteroids in the shower (the number of asteroids crossing a unit area perpendicular to the initial velocity) is n, how many of them will hit the Earth? You can ignore the effects of other bodies in the Solar system.

M07M.2 - Particle in an Ellipsoid

Problem

Consider a point mass m constrained to move without friction on the surface of an ellipsoid. There is no gravity in this problem. The coordinates of the mass can be parametrized by the following equations:

$$x = a \sin\theta \cos\phi \quad y = a \sin\theta \sin\phi \quad z = b \cos\theta.$$

a) Write the Lagrangian using $(\theta, \phi, \dot{\theta}, \dot{\phi})$ coordinates and derive the equations of motion.

b) Show that one period of the motion is given by

$$T(E, A) = 2 \int_{\theta_-}^{\theta_+} \frac{d\theta}{\sqrt{-V_{E,A}(\theta)}}$$

where

$$V_{E,A}(\theta) = \frac{ma^2A^2 - 2E\sin^2\theta}{m\sin^2\theta(a^2\cos^2\theta + b^2\cos^2\theta)}$$

and $\theta_\pm$ are the two roots of the equation $V_{E,A}(\theta) = 0$. Here E is the energy and A is the conserved quantity $A = \dot{\phi}\sin^2\theta$.

M07M.3 - Mass on a Massive String

Problem

A point mass M is freely hanging from a string of mass m and length l in the presence of gravity. The upper end of the string is held fixed. You can assume that $M \gg m$ so the tension in the string is approximately constant.

a) Write down the wave equation for small transverse oscillations of the string and the boundary conditions at the end of the string.

b) Determine the transcendental equation whose solutions give the oscillation frequencies of the normal modes on the string.

c) Approximately find the first two solutions of the transcendental equation for $M \gg m$ and give their physical interpretation.

M07E.1 - Point Charge an Conducting Sphere

Problem

A solid sphere of radius R is uncharged and isolated in space. A point charge q is slowly brought from far away and held a distance d from the center of the sphere.

a) What is the force between the point charge and the sphere?

b) What is the distribution of charges on the sphere?

M07E.2 - Noise in a Circular Ring

Problem

A circular ring of radius a is made from copper wire. The ring is held at a temperature T. The wire diameter is d and its electrical conductivity is σ.

a) What is the voltage noise across the ends of the wire if the ring is open? State your result in terms of root mean square voltage V_n in a frequency bandwidth Δf. Use V_n in subsequent parts if you are uncertain about its value.

b) Suppose the ends of the ring are shorted. What is the r.m.s. magnetic field noise in a bandwidth Δf at the center of the ring at very low frequencies?

c) Consider the r.m.s. magnetic noise in a narrow bandwidth Δf around a central frequency f. The magnetic field noise is constant up to some critical frequency f_c and drops as f^{-p} for frequencies much higher than f_c. There are two effects which are responsible for this decrease of the magnetic noise in the radio-frequency range, where the electrical conductivity σ is nearly constant. What are these effects? Give a rough estimate of f_c, which is approximately the same for both effects, and find the power p.

M07E.3 - Plane Wave in a Conductor

Problem

A plane electromagnetic wave with electric field E_0 and frequency ω is incident at normal incidence on a metal film with conductivity $\sigma, \varepsilon = \varepsilon_0$ and $\mu = \mu_0$.

a) Calculate the electric and magnetic fields as a function of distance x into the conductor.

b) Show that the energy lost by the electromagnetic wave in a small distance Δx inside the conductor is equal to the ohmic heat deposited by the electromagnetic wave in that distance.

M07Q.1 - Quantum Virial Theorem

Problem

Let $F(\boldsymbol{r}, \boldsymbol{p})$ be some function of position and momentum without explicit time dependence, $\partial F/\partial t = 0$.

a) If $|\psi_n\rangle$ is an eigenstate of a Hamiltonian H in the Schrödinger representation, show that

$$\frac{d}{dt}(\langle\psi_n|F|\psi_n\rangle) = 0.$$

b) Suppose

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}).$$

Show that

$$\langle\psi_n|\frac{\boldsymbol{p}^2}{2m}|\psi_n\rangle = \frac{1}{2}\langle\psi_n|\boldsymbol{r}\cdot\nabla V(\boldsymbol{r})|\psi_n\rangle.$$

c) Use this quantum-mechanical version of the virial theorem to estimate the fraction of the proton rest mass that is in the form of potential energy. The gluon-mediated force between two quarks is nearly independent of the distance between them. The rest mass of the quarks is much smaller than the mass of the proton, so strictly speaking one should use a relativistic version of the virial theorem. However, the non-relativistic version still gives approximately correct results, see "Relativistic virial theorem", Phys. Rev. Lett. **64**, 2733-2735 (1990).

M07Q.2 - Scattering

Problem

Consider elastic scattering from a 3-dimensional spherically symmetric square potential well of radius R:

$$V(\boldsymbol{r}) = \begin{cases} -V_0, & |\boldsymbol{r}| < R \\ 0, & |\boldsymbol{r}| \geq R \end{cases}$$

A beam of particles with mass m is impinging on the potential well with incident momentum p_0 parallel to the $\hat{z}$ axis.

a) Calculate the differential scattering cross-section as a function of the scattering angle θ in the Born approximation for $p_0 \ll \hbar/R$.

b) Sketch the differential cross-section as a function of the scattering angle θ and describe any prominent features.

c) Suppose now we have an infinite two-dimensional array of square wells in the x-y plane separated by distance $a \gg 2R$:

$$V_a(\boldsymbol{r}) = \sum_{j,k=1}^{\infty} V(\boldsymbol{r} - ja\hat{x} - ka\hat{y}).$$

Describe qualitatively the scattering pattern in this case and sketch the behavior of the cross-section, labeling the position of prominent features.

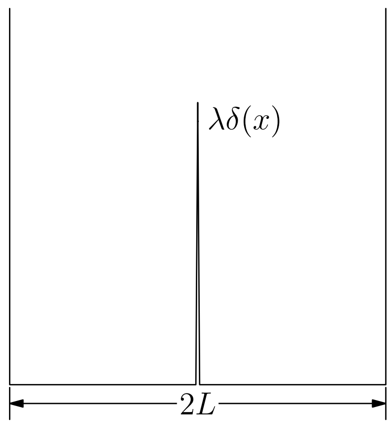
M07Q.3 - Delta Function in a Box

Problem

A particle of mass m is confined to a square potential well of width $2L$ with a delta function in the center:

$$V(x) = \begin{cases} \lambda\delta(x), & |x| < L \\ \infty, & |x| \geq L \end{cases}$$

In this problem λ is always large, $\lambda \gg \hbar^2/mL$.



a) What are the energies of the two lowest energy eigenstates, which are odd and even under parity, when $\lambda \to \infty$?

b) Now consider the case when λ is large but finite and find the energies of the lowest odd and even parity eigenstates to the lowest order in $1/\lambda$.

c) Suppose that at $t = 0$ the particle is localized on the right-hand side of the well and has the lowest possible energy. Calculate the probability of finding the particle on the left-hand side of the well as a function of time.

M07T.1 - Pions

Problem

a) The π meson is a spinless boson which can exist in three states of charge: π^+, π^0, and π^-. We assume its mass to be zero, which is a good approximation at high enough temperatures. We therefore are in the ultrarelativistic case where the dispersion relation (in units where $c = 1$) is given by $\varepsilon_p = |\vec{p}| = p$. Calculate (in units where $\hbar = 1$ and $k_B = 1$) the pressure $P_\pi(T)$ of a gas of π mesons at temperature T and chemical potential 0.

[You may need $\int_0^\infty dx\, x^3(e^x - 1)^{-1} = \pi^4/15$.]

b) The π mesons are made of "up" and "down" quarks and antiquarks, which are spin-1/2 fermions that can be considered massless, interacting via the exchange of "gluons", which are massless spin one bosons. At low temperatures the quarks and gluons don't exist as free particles, they are confined inside the mesons. As the temperature rises, a transition takes place that frees the quarks and gluons. We thus obtain a gas, assumed ideal, of quarks and gluons: the *quark-gluon* plasma. Taking into account "colour" charges, this gas has a total of 24 fermion (quark) degrees of freedom and 16 boson (gluon) degrees of freedom. Calculate the pressure of this ideal gas $P_{plasma}(T)$ for zero quark and zero gluon chemical potential.

[You may need $\int_0^\infty dx\, x^3(e^x + 1)^{-1} = 7\pi^4/120$.]

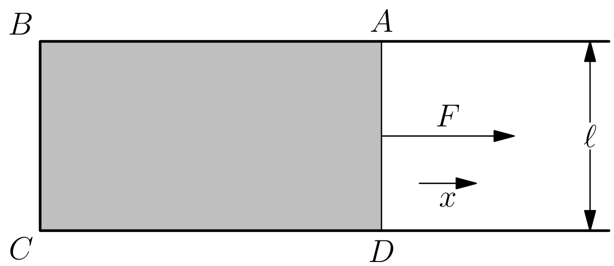
c) The above expression should be modified in order to take into account the effect of interactions: if the plasma is in a box of volume V one has to add to its free energy a volume contribution BV, where B is a positive constant. How is the expression for P_{plasma} modified?

d) Compare the expressions for the pressure of the π meson gas and the quark-gluon plasma. Show that at low temperature the meson phase is stable, while at high temperature the plasma phase is stable. Find the phase transition temperature T_c.

M07T.2 - Soap Film

Problem

A soap film is held by the four sides of a rectangle $ABCD$. The wire AD, of length ℓ, can be moved while kept parallel to BC allowing the film to be stretched. The strength of the force $\vec{F}$, applied in the positive x direction (*i.e.*, in the direction parallel to BA and CD), needed to keep AD fixed is $\sigma\ell$, where σ is the surface tension.



a) Denoting by $U(T, x)$ the internal energy of the soap film, where T is the temperature and $x = |AB|$, write the equation for TdS, where S is the entropy.

Now define the specific heat at constant length as

$$C_x = T\,\frac{\partial S}{\partial T}\bigg|_x .$$

For a wide range of temperatures close to room temperature, C_x is approximately constant (both with respect to T and x) and the surface tension varies linearly with T:

$$\sigma = \sigma_0(1 - a(T - T_0))$$

where σ_0, a and T_0 are positive constants.

b) We stretch the film by dx *quasi-statically* and at *constant temperature*. Calculate the corresponding infinitesimal increase of internal energy dU. Verify thet heat energy must be given to the film on order to maintain its constant temperature.

c) We now stretch the fillm *quasi-statically* and *adiabatically* by dx. Calculate the resulting temperature variation dT that accompanies this stretching. Is it positive or negative?

d) Sketch a reversible Carnot cycle (*i.e.*, a reversible cycle built out of two isothermal and two adiabatic transformations) on a σ-x diagram and indicate the x dependence of the adiabatic and isothermal curves.

M07T.3 - 3D Ising Model

Problem

The Ising model on a 3-dimensional square lattice with spin-1/2 particles is defined by the Hamiltonian

$$H = -J \sum_{\boldsymbol{i},\boldsymbol{j}} \sigma_{\boldsymbol{i}} \sigma_{\boldsymbol{i}+\boldsymbol{j}} - B \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}}$$

where $J > 0$, $\boldsymbol{i}$ labels sites of the 3-dimensional lattice, $\boldsymbol{j}$ runs over nearest neighbor sites in 3 dimensions and $\sigma_{\boldsymbol{i}}$ is equal to $+1$ or -1.

The Ising model is often solved using the *mean field approximation*, consisting of replacing the spin interaction Hamiltonian by the mean field interaction

$$H_m = -M \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}} - B \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}}$$

where M is a parameter fixed by the self-consistency condition to be $M = 6J\langle \sigma_{\boldsymbol{i}} \rangle$.

a) For the Hamiltonian H_m calculate the free energy, entropy, and $\langle \sigma_{\boldsymbol{i}} \rangle$ at temperature T.

b) Show that for $B = 0$ at low temperature a self-consistent solution with $\langle \sigma_{\boldsymbol{i}} \rangle \neq 0$ has a lower free energy than a solution with $\langle \sigma_{\boldsymbol{i}} \rangle = 0$.

c) Find the critical temperature T_c above which the spontaneous magnetization vanishes at zero external field B.

d) How can one build a refrigerator using the spins as the working substance? Describe qualitatively how one can efficiently cool a substance by manipulating the spin degrees of freedom which obey this mean field theory.