1 May 2006, Thermodynamics, Problem 2

1.1 (a)

$$Z_1 = \int e^{-(p_x^2+p_y^2+p_z^2)/2MT} e^{-p_\theta^2/2IT} e^{-p_\phi^2/2Isin^2\theta T} e^{\mu E cos\theta/T} \frac{d\phi\, d\theta\, dp_x\, dp_y\, dp_z\, dx\, dy\, dz\, dp_\theta\, dp_\phi}{(2\pi)^5}$$

$$Z_1 = (MT)^{3/2}(IT)\int_0^\pi sin\theta e^{\mu E cos\theta/T}\frac{V\, d\theta}{(2\pi)^{3/2}} = \left(\frac{MT}{2\pi}\right)^{3/2} ITV\frac{T}{\mu E}\int_{-\mu E/T}^{\mu E/T} e^x\, dx$$

$$Z_1 = \left(\frac{MT}{2\pi}\right)^{3/2}\frac{2IT^2V}{\mu E}sinh(\mu E/T)$$

$$Z = \frac{Z_1^N}{N!}$$

$$F_N = -TlnZ = -T(NlnZ_1 - NlnN + N) = Nk_BT\left[lnN - 1 - ln\left(\left(\frac{MT}{2\pi}\right)^{3/2}\frac{2IT^2V}{\mu E}sinh(\mu E/T)\right)\right] \tag{1}$$

1.2 (b)

The dipole moment per unit volume is going to be the average dipole moment of each particle, times the particle density. So one way to compute the polarization is:

$$< cos\theta >= \int e^{-(p_x^2+p_y^2+p_z^2)/2MT} e^{-p_\theta^2/2IT} e^{-p_\phi^2/2Isin^2\theta T} e^{\mu E cos\theta/T} \frac{cos\theta\, d\phi\, d\theta\, dp_x\, dp_y\, dp_z\, dx\, dy\, dz\, dp_\theta\, dp_\phi}{Z_1(2\pi)^5} =$$

$$= \frac{T}{\mu Z_1}\frac{\partial Z_1}{\partial E} = \frac{T}{\mu}\frac{\partial lnZ_1}{\partial E} = \frac{T}{\mu}\left[\frac{\mu cosh(\mu E/T)}{Tsinh(\mu E/T)} - \frac{1}{E}\right] = coth(\mu E/T) - \frac{T}{\mu E}$$

$$\boldsymbol{P_N} = \frac{N}{V} < \boldsymbol{\mu} >= \frac{N}{V}\mu < cos\theta > \hat{z} = \frac{N\mu\hat{z}}{V}\left[coth(\mu E/T) - \frac{T}{\mu E}\right] \tag{2}$$

$$cothx = \frac{coshx}{sinhx} \approx \frac{1 + x^2/2}{x} \quad x << 1$$

$$\boldsymbol{P_N} = \frac{N\mu\hat{z}}{V}\left(\frac{\mu E}{2T}\right) \quad \frac{\mu E}{T} << 1 \tag{3}$$

$$(\epsilon - 1)E = 4\pi P$$

$$\epsilon = \frac{4\pi P}{E} + 1 = 1 + \frac{4\pi}{E}\frac{N\mu}{V}\left[coth(\mu E/T) - \frac{T}{\mu E}\right] \tag{4}$$

$$\epsilon = 1 + \frac{2\pi N\mu^2}{VT} \quad \frac{\mu E}{T} << 1 \tag{5}$$