1 May 2006, Quantum Mechanics, Problem 3

1.1 (a)

Express the time-evolved state as:

$$\Psi(x_1, x_2, t) = \sum_n \alpha_n(t) e^{-iE_n(t-t_0)} \psi_n(x_1, x_2)$$

where the coefficients potentially depend on time, and the ψ_n's are eigenstates of the unperturbed Hamiltonian. t_0 is the time at which we are told the system is in the ground state of the harmonic oscillator. Then applying time-dependent perturbation theory gives:

$$\dot{\alpha_n} = -i < \psi_n | V_{int} | \psi_i > e^{i(E_n - E_i)(t-t_0)}$$

$$\alpha_n(t) = \delta_{ni} - i \int_{t_0}^{t} < \psi_n | V_{int} | \psi_i > e^{i(E_n - E_i)(t'-t_0)}\, dt'$$

where i denotes the initial state. This way we can calculate the coefficient of the state we care about, which we call ψ_1:

$$\psi_1 = |\psi_1(x_2) > |p_f(x_1) >$$

$$\alpha_1 = -i \int_{t_0}^{t} < \psi_1 | V_{int} | \psi_i > e^{i(E_1 - E_i)(t'-t_0)}\, dt'$$

$$< \psi_1 | V_{int} | \psi_i >= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{2m\omega} \left(\frac{m\omega}{\pi}\right)^{1/4} x_2 e^{-m\omega x_2^2/2} e^{-ip_f x_1} \lambda \delta(x_1 - x_2) \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-m\omega x_2^2/2} e^{ip_i x_1}\, dx_1\, dx_2$$

$$= \sqrt{2m\omega} \lambda e^{-(p_i - p_f)^2/4m\omega} \frac{i(p_i - p_f)}{2m\omega}$$

$$\alpha_1 = \lambda e^{-(p_i - p_f)^2/4m\omega} \frac{(p_i - p_f)}{\sqrt{2m\omega}} e^{-i(E_1 - E_i)t_0} \int_{t_0}^{t} e^{i(E_1 - E_i)t'}\, dt'$$

Now notice that:

$$E_i = \frac{p_i^2}{2m} + \frac{\omega}{2}$$

$$E_1 = \frac{p_f^2}{2m} + \frac{3\omega}{2}$$

$$\alpha_1 = \lambda e^{-(p_i - p_f)^2/4m\omega} \frac{(p_i - p_f)}{\sqrt{2m\omega}} e^{-i(\omega + \frac{p_f^2}{2m} - \frac{p_i^2}{2m})t_0} \int_{t_0}^{t} e^{i(\omega + \frac{p_f^2}{2m} - \frac{p_i^2}{2m})t'}\, dt'$$

Since the problem doesn't say "at time t" and instead it says "ends up in the first excited state", we'll assume they mean that $t_0 \to -\infty$ and $t \to \infty$, so:

$$\alpha_1 = \lambda e^{-(p_i - p_f)^2/4m\omega} \frac{(p_i - p_f)}{\sqrt{2m\omega}} e^{-i(\omega + \frac{p_f^2}{2m} - \frac{p_i^2}{2m})t_0} 2\pi \delta(\omega + \frac{p_f^2}{2m} - \frac{p_i^2}{2m})$$

The only allowed value of final momentum, according to the delta function, is:

$$p_f = \sqrt{p_i^2 - 2m\omega}$$

$$\mathcal{P} = \lambda^2 e^{-(p_i - \sqrt{p_i^2 - 2m\omega})^2/2m\omega} \frac{(p_i - \sqrt{p_i^2 - 2m\omega})^2}{2m\omega} 4\pi^2 \tag{1}$$