1 May 2006, Quantum Mechanics, Problem 1

1.1 (a)

The Schrdinger equation is:

$$-\frac{1}{2m}\Psi'' - \frac{A}{x}\Psi = E\Psi$$
$$\Psi'' + \frac{2mA}{x}\Psi + 2mE\Psi = 0$$

I now introduce an extra term, which I can do freely as long as I then set its multiplicative parameter l to 0:

$$\Psi'' + \frac{2mA}{x}\Psi + 2mE\Psi - \frac{l(l+1)}{r^2}\Psi = 0$$

Notice that this is the same equation that we had for the radial part of the hydrogen atom solution, with $A = e^2/(4\pi\epsilon_0)$. In that case, we found that the solution with the lowest value of E was:

$$\Psi = Bxe^{-a*x}$$

where B and a are (so far) undetermined constants. This actually solves the equation with:

$$l = 0$$
$$a = Am$$
$$\Psi(0) = 0$$
$$\lim_{x\to\infty}\Psi(x) = 0$$
$$E = -\frac{A^2 m}{2\hbar^2} \tag{1}$$

1.2 (b)

$$<\Psi|\Psi> = \frac{B^2}{(2Am)^3}2! = 1 \rightarrow B^2 = \frac{(2Am)^3}{2}$$
$$<x> = <\Psi|x|\Psi> = \int_0^\infty B^2 e^{-2Amx}x^3\,dx = \frac{B^2}{(2Am)^4}3! = \frac{3\hbar^2}{2Am} \tag{2}$$

I think the "well-known result" they were referring to was:

$$\int_0^\infty e^{-x}x^n\,dx = n!$$