1 May 2006, Quantum Mechanics, Problem 1

1.1 (a)

The Schrödinger equation is:

\[-\frac{1}{2m} \Psi'' - \frac{A}{x} \Psi = E \Psi\]
\[\Psi'' + \frac{2mA}{x} \Psi + 2mE \Psi = 0\]

I now introduce an extra term, which I can do freely as long as I then set its multiplicative parameter \(l\) to 0:

\[\Psi'' + \frac{2mA}{x} \Psi + 2mE \Psi - \frac{l(l+1)}{r^2} \Psi = 0\]

Notice that this is the same equation that we had for the radial part of the hydrogen atom solution, with \(A = e^2/(4\pi\epsilon_0)\). In that case, we found that the solution with the lowest value of \(E\) was:

\[\Psi = Bxe^{-ax}\]

where \(B\) and \(a\) are (so far) undetermined constants. This actually solves the equation with:

\[l = 0\]
\[a = Am\]
\[\Psi(0) = 0\]
\[\lim_{x \to \infty} \Psi(x) = 0\]
\[E = \frac{A^2m}{2\hbar^2}\] (1)

1.2 (b)

\[<\Psi|\Psi> = \frac{B^2}{(2Am)^3}2! = 1 \rightarrow B^2 = \frac{(2Am)^3}{2}\]
\[<x> = <\Psi|x|\Psi> = \int_0^\infty B^2 e^{-2Ax^3}dx = \frac{B^2}{(2Am)^4}3! = \frac{3\hbar^2}{2Am}\] (2)

I think the "well-known result" they were referring to was:

\[\int_0^\infty e^{-x}x^n\ dx = n!\]