1 May 2006, Mechanics, Problem 3

Suppose we place the spring horizontally on a table and we stretch it by a length \( r \). Then the tension in the spring will be \( Kr \). We could also visualize this as a chain of little springs of length \( dx \) (there will be \( \frac{L}{dx} \) of them, each one stretching by \( \frac{r}{L} \)). The spring constant of each of these little springs is denoted by \( k_{dx} \). Because the tension is constant throughout the spring, we can write:

\[
k_{dx} \frac{r}{L} = Kr \rightarrow k_{dx} = \frac{KL}{dx}
\]

Now label each point by its position \( x \) along the spring when it’s placed horizontally, and then hang it vertically. The equation for stability of each section of length \( dx \) is:

\[
T(x + dx) + \rho dx g = T(x) \rightarrow \frac{dT}{dx} = -\rho g
\]

And we need to match this to the value of the tension. In each little piece of length \( dx \), we call \( s(x) \) the displacement from the equilibrium position:

\[
T(x) = k_{dx} \left( s(x + dx) - s(x) \right) = KL \frac{ds}{dx}
\]

\[
KLs'' = -\rho g
\]

\[
s(x) = -\frac{\rho g}{2KL} x^2 + Bx + C
\]

\[
s(0) = C = 0
\]

\[
s'(x) = -\frac{\rho g}{KL} x + B
\]

\[
T(L) = KL \left( -\frac{\rho g}{K} + B \right) = 0 \rightarrow B = \frac{\rho g}{K}
\]

Finally, the position with respect to the top is the original displacement plus the displacement from equilibrium:

\[
y(x) = x + s(x) = x - \frac{\rho g}{2KL} x^2 + \frac{\rho g}{K} x
\]

(1)