1 May 2006, Mechanics, Problem 3

Suppose we place the spring horizontally on a table and we stretch it by a length r. Then the tension in the spring will be Kr. We could also visualize this as a chain of little springs of length dx (there will be $\frac{L}{dx}$ of them, each one stretching by $\frac{rdx}{L}$. The spring constant of each of these little springs is denoted by k_{dx}. Because the tension is constant throughout the spring, we can write:

$$k_{dx}\frac{rdx}{L} = Kr \rightarrow k_{dx} = \frac{KL}{dx}$$

Now label each point by its position x along the spring when it's placed horizontally, and then hang it vertically. The equation for stability of each section of length dx is:

$$T(x+dx) + \rho dxg = T(x) \rightarrow \frac{dT}{dx} = -\rho g$$

And we need to match this to the value of the tension. In each little piece of length dx, we call s(x) the displacement from the equilibrium position:

$$T(x) = k_{dx}\left(s(x+dx) - s(x)\right) = KL\frac{ds}{dx}$$

$$KLs'' = -\rho g$$

$$s(x) = -\frac{\rho g}{2KL}x^2 + Bx + C$$

$$s(0) = C = 0$$

$$s'(x) = -\frac{\rho g}{KL}x + B$$

$$T(L) = KL\left(-\frac{\rho g}{K} + B\right) = 0 \rightarrow B = \frac{\rho g}{K}$$

Finally, the position with respect to the top is the original displacement plus the displacement from equilibrium:

$$y(x) = x + s(x) = x - \frac{\rho g}{2KL}x^2 + \frac{\rho g}{K}x \quad (1)$$