1 May 2006, Mechanics, Problem 2

1.1 (a)

$$\mathcal{L} = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - mgh$$

where h is the vertical position of the ball.

$$\boldsymbol{x} = \langle -x - lsin\theta, -y - lcos\theta \rangle$$

$$\dot{\boldsymbol{x}} = \left\langle -\dot{x} - \dot{l}sin\theta - lcos\theta\dot{\theta}, -\dot{y} - \dot{l}cos\theta + lsin\theta\dot{\theta} \right\rangle = \left\langle -x'\dot{\theta} - l'\dot{\theta}sin\theta - lcos\theta\dot{\theta}, -y'\dot{\theta} - l'\dot{\theta}cos\theta + lsin\theta\dot{\theta} \right\rangle$$

where the primes denote differentiation with respect to θ.

$$\dot{\boldsymbol{x}}^2 = \left(x'\dot{\theta} + l'\dot{\theta}sin\theta + lcos\theta\dot{\theta}\right)^2 + (-y'\dot{\theta} - l'\dot{\theta}cos\theta + lsin\theta\dot{\theta})^2 =$$

$$= \dot{\theta}^2(x'^2 + l'^2 + l^2 + 2x'l'sin\theta + 2x'lcos\theta + y'^2 + 2y'l'cos\theta - 2y'lsin\theta)$$

$$l_0 - l = \int_0^\theta ds = \int_0^\theta \sqrt{dx^2 + dy^2} = \int_0^\theta \sqrt{x'^2 + y'^2}\, d\theta \rightarrow l = l_0 - \int_0^\theta \sqrt{x'^2 + y'^2}\, d\theta \qquad (1)$$

$$l' = -\sqrt{x'^2 + y'^2}$$

$$l'^2 = x'^2 + y'^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{\theta}^2 \left(2(x'^2 + y'^2) + l^2 + 2l(x'cos\theta - y'sin\theta) - 2\sqrt{x'^2 + y'^2}(x'sin\theta + y'cos\theta)\right) + mg(y + lcos\theta)$$

$$\frac{dx}{dy} = tan\theta = \frac{x'}{y'}$$

$$\sqrt{x'^2 + y'^2} = y'\sqrt{1 + tan^2\theta} = y'sec\theta$$

$$\mathcal{L} = \frac{1}{2}m(\dot{\theta}l)^2 + mg(y + lcos\theta) \qquad (2)$$

1.2 (b)

We need the lagrangian to be of the form of a simple harmonic oscillator:

$$\mathcal{L} = c_1\dot{z}^2 - c_2 z^2 \qquad (3)$$

Without change to the physical results, we can multiply the Lagrangian by $m/2c_1$, and define the new constant in front of z^2 as c:

$$\mathcal{L} = \frac{m}{2}\left(\dot{z}^2 - cz^2\right)$$

$$\dot{z} = l\dot{\theta} \rightarrow z' = l$$

$$-cz^2 = 2g(y + lcos\theta)$$

$$-2czl = -2czz' = 2g(y' + l'cos\theta - lsin\theta)$$

$$y' + l'cos\theta = y' - \sqrt{y'^2 + x'^2}cos\theta = y' - y'sec\theta cos\theta = 0$$

$$cz = g(sin\theta)$$

$$l = z' = \frac{gcos\theta}{c}$$

$$l' = -y'sec\theta \rightarrow y' = -l'cos\theta = \frac{gsin(2\theta)}{2c}$$

$$y(\theta) = -\frac{gcos(2\theta)}{4c} + \frac{g}{4c} \tag{4}$$

$$x' = y'tan\theta = \frac{g}{c}\frac{1 - cos(2\theta)}{2}$$

$$x(\theta) = \frac{g}{2c}\left[\theta - \frac{sin(2\theta)}{2}\right] \tag{5}$$