1 May 2006, Mechanics, Problem 1

1.1 (a)

The easiest way to find the Lagrangian (for me, that is) is to imagine ”painting” a red line vertically downward when the system is at rest, and then rotate the big sphere to the right. The red line moves clockwise by an angle y/R, where y is the horizontal distance that the center of the sphere moves. Now, if the point mass is making an angle θ with respect to the vertical, as shown in the picture, then it is making an angle $(\theta + y/R)$ with respect to the red line.

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}\frac{2MR^2}{3}\left(\frac{\dot{y}}{R}\right)^2 + \frac{1}{2}mR^2\left(\frac{\dot{y}}{R} + \dot{\theta}\right)^2 + mgRcos\theta \tag{1}$$

where $\frac{2MR^2}{3}$ is the moment of inertia of the sphere.

1.2 (b)

$$\mathcal{L} \approx \frac{5}{6}M\dot{y}^2 + \frac{1}{2}mR^2\left(\frac{\dot{y}}{R} + \dot{\theta}\right)^2 + mgR(1 - \theta^2/2)$$

$$\frac{5M\ddot{y}}{3} + mR\left(\frac{\ddot{y}}{R} + \ddot{\theta}\right) = 0$$

$$mR^2\left(\frac{\ddot{y}}{R} + \ddot{\theta}\right) = -mgR\theta$$

$$\ddot{y} = -(g\theta + R\ddot{\theta})$$

$$-g\theta\left(m + \frac{5M}{3}\right) = \frac{5MR\ddot{\theta}}{3}$$

$$\omega = \sqrt{\frac{g\theta}{R}\left(\frac{3m}{5M} + 1\right)} \tag{2}$$