1 May 2006, Electromagnetism, Problem 2

1.1 (a)

The initial flux going through the loop is:

$$\Phi_B = LI_0$$

and it's going into the plane of the picture. After we turn on the current, there will be an induced magnetic field:

$$B = \frac{\mu_0 I}{2\pi\rho}\hat{z}$$

The induced magnetic flux through the loop is:

$$\Phi_B = \int_{-l}^{l}\int_{d-l}^{d+l} \frac{\mu_0 I}{2\pi\rho}\, dy\, dx = \frac{\mu_0 l I}{\pi} ln\left(\frac{d+l}{d-l}\right)$$

and the direction is coming out of the plane of the picture, by Lenz' law. At the same time, if we call the final current in the loop I_f, we have a total final flux:

$$\Phi_B = LI_f - \frac{\mu_0 l I}{\pi} ln\left(\frac{d+l}{d-l}\right)$$

Because the loop is superconducting, the flux must be conserved. If it changed, there would be an induced electromotive force, which in turn would generate an infinite current because the resistance of a superconducting loop is 0. Thus, we can write:

$$LI_0 = LI_f - \frac{\mu_0 l I}{\pi} ln\left(\frac{d+l}{d-l}\right)$$

This way we can find the final current (assumed to be in the same direction of I_0):

$$I_f = I_0 + \frac{\mu_0 l I}{\pi L} ln\left(\frac{d+l}{d-l}\right)$$

In order for the force to be attractive, I and I_f have to have the same direction (where the direction of I_f is defined along the bottom piece of the loop). Because of the sign convention, therefore, we need:

$$sign(I_0) = -sign(I)$$

Suppose we say I_f is negative. Then in order for the equation to hold, I must be negative too, and then they will have the same direction. So then we say I_f is positive, and we look for solution with negative I. But it must be small enough that the first term dominates, in order for I_f to stay positive:

$$-I_0 \frac{\pi L}{\mu_0 l \ln(d + l/(d-l))} < I < 0 \tag{1}$$

1.2 (b)

The force is:

$$F_{repulsive} = I_f 2l \frac{\mu_0 I}{2\pi(d-l)} - I_f 2l \frac{\mu_0 I}{2\pi(d+l)}$$

$$F_{attractive} = -I_f l \frac{\mu_0 I}{\pi(d-l)} + I_f l \frac{\mu_0 I}{\pi(d+l)} = \frac{I_f l \mu_0 I}{\pi}\left(\frac{-2l}{d^2-l^2}\right) = \frac{l\mu_0 I}{\pi}\left(\frac{-2l}{d^2-l^2}\right)\left[I_0 + \frac{\mu_0 l I}{\pi L}\ln\left(\frac{d+l}{d-l}\right)\right]$$

$$F_{attractive} = \frac{l\mu_0 i}{\pi}\left(\frac{2l}{d^2-l^2}\right)\left[I_0 - \frac{\mu_0 l i}{\pi L}\ln\left(\frac{d+l}{d-l}\right)\right]$$

where $i \equiv -I$. We have to maximize this force with respect to i.

$$i_{max} = \frac{I_0 \pi L}{2\mu_0 l \ln\left(\frac{d+l}{d-l}\right)} \quad (2)$$

1.3 (c)

$$F_{max} = \left(\frac{l}{d^2-l^2}\right)\frac{I_0^2 L}{2\ln\left(\frac{d+l}{d-l}\right)} \quad (3)$$