1 May 2006, Electromagnetism, Problem 1

1.1 (a)

We have to solve $\nabla^2 V = 0$ in spherical coordinates, with azimuthal symmetry. This solution was found before, in electromagnetism classes, etc... It is:

$$V = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(cos\theta)$$

where P_l are the Legendre polynomials. We need to match the boundary conditions:

$$\boldsymbol{E}_{r\to\infty} = E_0 \hat{z}$$
$$Q = 0$$
$$\left(\frac{\partial V}{\partial \theta}\right)_{r=R} = 0$$

The first condition leads us to:

$$V_{r\to\infty} = -E_0 z = -E_0 r cos\theta$$
$$V_{r\to\infty} = \sum_{l=0}^{\infty} (A_l r^l) P_l(cos\theta)$$

Noting that $P_0(x) \sim 1$ and $P_1(x) \sim cos\theta$, so l=0 and l=1 are the only allowed modes, and $A_1 = -E_0$:

$$V = A_0 + B_0 r^{-1} + (-E_0 r + B_1 r^{-2}) cos\theta$$

We can set $A_0 = 0$ because the physics are invariant under a constant added to the potential. Now enforce the condition that the component parallel to the surface of the sphere, at the sphere's surface, must be 0:

$$\frac{\partial V}{\partial \theta} = -sin\theta(-E_0 R + B_1 R^{-2}) = 0 \to B_1 = E_0 R^3$$
$$\frac{\partial V}{\partial r} = -B_0 r^{-2} + cos\theta(-E_0 - 2B_1 r^{-3})$$
$$\sigma = \epsilon_0[B_0 R^{-2} + cos\theta(E_0 + 2B_1 R^{-3})]$$

$$Q = \int_0^{2\pi}\int_0^{\pi} \sigma R^2 sin\theta\, d\theta\, d\phi = R^2 \epsilon_0 2\pi \int_0^{\pi} [B_0 R^{-2} + cos\theta(E_0 + 2B_1/R^3)] sin\theta\, d\theta = 2\pi\epsilon_0 R^2 B_0 R^{-2} 2$$

$$Q = 0 \to B_0 = 0$$

$$V = (-E_0 r + E_0 R^3 r^{-2}) cos\theta \tag{1}$$

1.2 (b)

We already found the charge density in part a. Plugging in the values we found for B_0 and B_1:

$$\sigma = \epsilon_0 cos\theta(E_0 + 2E_0)cos\theta = 3\epsilon_0 E_0 cos\theta \tag{2}$$