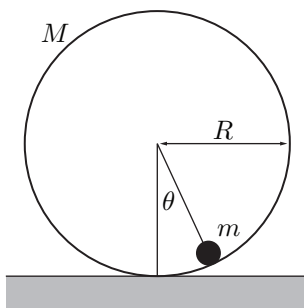


M06M.1 - Point Mass in a Sphere

Problem

A point mass m slides without friction inside of a hollow sphere of mass M and radius R , that rolls without slipping over a horizontal surface.

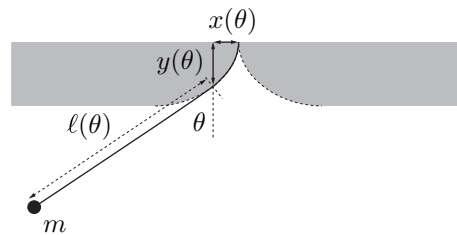


- Find the Lagrangian of this system. Assume that the velocities of the point mass and the sphere are always in the plane of the paper.
- Consider small amplitude oscillations about the equilibrium position. Express the oscillation frequency in terms of m, M, R and the gravitational constant g .

M06M.2 - Huygens' Pendulum (M92M.1)

Problem

To compensate for the fact that the period of a simple pendulum depends on the amplitude of oscillation, the 17th century Dutch physicist Christian Huygens devised the following setup, depicted in the figure below. It shows a simple pendulum consisting of a mass m and a string of length ℓ_0 whose motion is constrained by a cusp shaped piece of wood. The problem is to determine the shape of the wooden surface so that the period of the pendulum is independent of the amplitude.



- Parametrize the shape of the surface by $x(\theta)$ and $y(\theta)$, as indicated in the figure. Write the Lagrangian for the pendulum.
- What property must the Lagrangian have in order for the period of oscillation to be independent of the amplitude? Find the required shape $(x(\theta), y(\theta))$.

M06M.3 - Hanging Spring

Problem

A spring has spring constant K , unstretched length L , and mass per unit length ρ (when unstretched). The spring is suspended from one end in a constant gravitational field, g , and stretches under its own weight. For a point whose distance from the upper end of the spring is x when unstretched, find its distance $y(x)$ from the upper end when the spring is stretched.

M06E.1 - Metal Sphere in a Uniform Electric Field

Problem

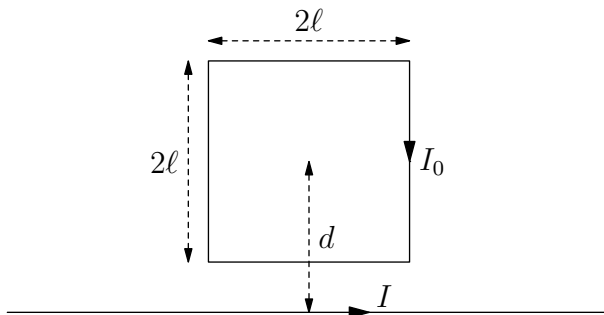
An uncharged metal sphere of radius R is placed inside an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$.

- a) Find the electrostatic potential in the region outside the sphere.
- b) Find the induced charge density on the surface of the sphere.

M06E.2 - Superconducting Loop

Problem

A current I_0 flows in a superconducting square loop of side 2ℓ and self-inductance L . An infinite wire, initially carrying no current, is in the plane of the loop at a distance $d > \ell$ from its center. When a current I is switched on in the wire, in the direction as indicated in the figure, a force between the loop and wire results.



- Find the range of values of I for which the force is attractive.
- For which value of I is the attractive force a maximum?
- Calculate the maximum attractive force.

M06E.3 - Radiation from a Falling Electron

Problem

An electron is released from rest at a large distance r_0 from a nucleus of charge Ze and then “falls” towards the nucleus. From what follows, assume the electron’s velocity is such that $v \ll c$ and the radiation reaction on the electron is negligible.

- a) What is the angular distribution of the emitted radiation?
- b) How is the emitted radiation polarized?
- c) What is the radiated power as a function of the separation between the electron and the nucleus?

M06Q.1 - One-dimensional Wave Function

Problem

A particle of mass m moves in one dimension to the right of a wall at $x = 0$ in the potential

$$V(x) = -\frac{A}{x}$$

where A is a given positive parameter.

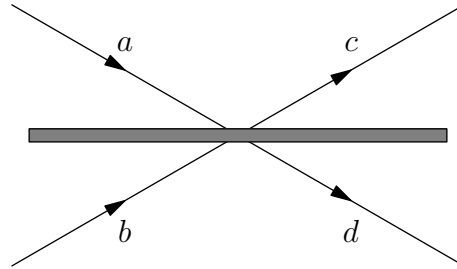
- a) Find the ground state energy.
- b) Find the position expectation value, $\langle x \rangle$, for the ground state.

There is no need to derive whatever “well known” results you find applicable here.

M06Q.2 - EPR Beam Splitter

Problem

In this problem, we consider two-photon interference at a beam splitter. The aim is to that, with a suitable measurement, the two photons can be projected onto an EPR-pair state after leaving the beam splitter. The beam splitter has two spatial input modes $|a\rangle$ and $|b\rangle$, corresponding to the two sides of a semi-transparent mirror:



Initially, photon 1 is horizontally polarized and arrives along the direction $|a\rangle$,

$$|\psi_i\rangle_1 = |\leftrightarrow\rangle_1 |a\rangle_1$$

while photon 2 is vertically polarized and arrives along the other direction $|b\rangle$:

$$|\phi_i\rangle_2 = |\updownarrow\rangle_2 |b\rangle_2.$$

The beam splitter is a semi-reflecting mirror, and maps the two spatial input states to a linear sum of a reflected and a transmitted state:

$$\begin{aligned} |a\rangle &\rightarrow \frac{1}{\sqrt{2}}(i|c\rangle + |d\rangle) \\ |b\rangle &\rightarrow \frac{1}{\sqrt{2}}(|c\rangle + i|d\rangle) \end{aligned}$$

The factor i corresponds to a phase jump upon reflection at the semi-transparent mirror.

- Write the output state $|\psi_f\rangle_i$ of photon i after it leaves the beam splitter.
 - Now write the total output state of the two photons. Remember: photons are bosons!
- A measurement shows that the two photons leave on opposite sides from the beam splitter.
- Give the probability that a position measurement of the photons yields this outcome.
 - What is the polarization state of the two photons after this measurement?
 - Write the density matrix that describes the polarization state of one of the photons.

M06Q.3 - Two Interacting Particles

Problem

Consider two particles of mass m moving in one dimension. Particle 1 moves freely, while particle 2 experiences a harmonic potential $V(x_2) = \frac{1}{2}m\omega^2x_2^2$. The two particles interact via a delta function potential

$$V_{int}(x_{12}) = \lambda\delta(x_{12}),$$

with $x_{12} \equiv x_1 - x_2$. Particle 2 starts in the ground state $|\psi_0\rangle$, and particle 1 comes in from the left in a momentum eigenstate $|p_i\rangle$. Compute the transition probability \mathcal{P}_{01} that particle 2 ends up in the first excited state $|\psi_1\rangle$, to leading order for small λ .

M06T.1 - Interacting Particles on a Line

Problem

Consider a system of N classical particles on a line with Hamiltonian

$$H = \frac{p_1^2}{2m} + U_1(x_1) + \sum_{i=2}^N \frac{p_i^2}{2m} + U(x_i - x_{i-1}).$$

The potential between neighboring particles is of the form:

$$U(y) = \begin{cases} +\infty, & \text{if } y < 0; \\ -U_0, & \text{if } 0 \leq y \leq a; \\ 0, & \text{if } a < y. \end{cases}$$

Here both U_0 and a are positive. A constant force f is applied to the rightmost particle $i = N$.

- Compute the mean length, $\langle x_N \rangle$, of the system as a function of N, T , and f .
- Obtain the high and low temperature limits of the result from part a).

M06T.2 - Diatomic Gas in an Electric Field (J07T.2)

Problem

The Hamiltonian for a diatomic molecule with constant dipole moment μ in a homogeneous electric field $\vec{E} \equiv (0, 0, E)$ is:

$$H = \frac{1}{2M}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2I}p_\theta^2 + \frac{1}{2I \sin^2 \theta}p_\phi^2 - \mu E \cos \theta$$

(M = mass of molecule, I = moment of inertia, and (r, θ, ϕ) are polar coordinates). Consider an ideal gas of N such *classical* molecules in a volume V , using *Boltzmann statistics*.

- a) Compute the free energy $F_N(T, V, E)$.
- b) Compute the dipole moment per unit volume (“polarization”), $\mathbf{P}_N(T, V, E)$, of the gas and evaluate the dielectric constant ϵ in the limit $\mu E \ll k_B T$.

[Recall: $\epsilon \mathbf{E} = \mathbf{E} + 4\pi \mathbf{P}$.]

M06T.3 - Boson Surface Absorption

Problem

Consider a 3-dimensional gas of (spinless, non-relativistic) bosons at pressure P and temperature T . The bosons can be absorbed onto a (2-dimensional) surface layer, where they are bound with energy $-\epsilon_0 < 0$, but retain their translational degrees of freedom in 2 dimensions. The (ideal) 3D gas is in equilibrium with the (ideal) 2D adsorbed gas. Treating the 3D gas classically, but the 2D (adsorbed) gas quantum mechanically, compute the surface density in the layer as a function of P and T .

(You may need: $\int \frac{dx}{ae^x+1} = \ln \frac{e^x}{1+ae^x}$.)