1 May 2005, Thermodynamics, Problem 1

1.1 (a)

The equation for the specific heat capacity at constant pressure is [1]:

$$c_P = \frac{1}{m}\left(\left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P\right)$$

Plugging in $dU = -PdV + TdS$, one can obtain:

$$c_P = \frac{T}{m}\left(\frac{\partial S}{\partial T}\right)_P$$

Solve for the derivative and then integrate with respect to T to obtain:

$$s = \frac{49.9T^3}{3\Theta^3} + g(P)$$

where g(P) is some arbitrary function of P. However, s(0) must be 0 by the third law of thermodynamics [1]. Thus,

$$s = \frac{49.9T^3}{3\Theta^3} \tag{1}$$

The internal energy cannot be obtained by any straightforward means. One way is to perform a full computation of the form:

$$U = \int_0^{\omega_D} g(\omega)\frac{\omega}{e^{\omega/T}-1}\,d\omega$$

but that is rather long and not so illuminating. The other way is to assume that at low temperature, the heat capacities match, because the change in volume as we cool or heat something is negligible, and therefore:

$$u = \int c_P\,dT = \frac{49.9T^4}{4\Theta^3}$$

$$s = \frac{4}{3}u \tag{2}$$

1.2 (b)

This is obtained by basic chemistry:

$$E = m\int_{4K}^{20K} c_P dT = \frac{4990}{4*92^3}(20^4 - 4^4)J \approx 256J \tag{3}$$

1.3 (c)

The maximum efficiency is obtained with a Carnot cycle. However, there are two differences with the most elementary Carnot cycle:

(i) it is operated in reverse: the cold gets colder and the hot gets hotter, by the input of work;

(ii) the temperature of the "cold reservoir" is changing, so the cycle must be treated differentially.

$$dQ_{in} - dQ_{out} + dW = 0 \quad Conservation\, of\, energy \tag{4}$$

$$dQ_{in} = -dQ_{argon} = -mc_P dT \quad Conservation\, of\, energy \tag{5}$$

$$\frac{dQ_{out}}{T_{room}} = \frac{dQ_{in}}{T_{argon}} \quad Reversibility \tag{6}$$

The first equation is conservation of energy in the cycle, while the second equation is conservation of energy in the part of the cycle that involves heat exchange between the solid argon and the gas. Use (4) to plug into (6), solve for dQ_{in} and plug in (5). Then integrate over temperature between 20 K and 4 K, to obtain:

$$W_{in} = \frac{49900 * m}{\Theta^3}\left(4^3\left(\frac{4}{4} - \frac{20}{3}\right) + \frac{20^4}{12}\right) J \approx 83J \tag{7}$$

References

[1] D. V. Schroeder, *An Introduction to Thermal Physics*. Addison Wesley Longman, 2000.