May 2005 #2 (QM)

m, E $\qquad E = \frac{\hbar^2 k^2}{2m}$

$V = vR\delta(r-R)$

a. S-wave scattering cross section σ_0

For the $l=0$ term (0 angular momentum),

$\sigma_0 = \dfrac{4\pi}{k^2}\sin^2\delta_l$ $\qquad \delta_l$ is the phase shift in the wave,

$R_0(r) \xrightarrow[r\to\infty]{} = \dfrac{A\sin[kr+\delta_l]}{r}$

• Solve the Schrödinger Equation to find the phase shift

$R_{0>} = Aj_0(kr) + Bn_0(kr) = \dfrac{A\sin(kr)}{kr} - \dfrac{B\cos(kr)}{kr} \qquad r>R$

$R_{0<} = Cj_0(kr) = \dfrac{C\sin(kr)}{kr} \qquad r<R$

Two Boundary conditions: ① ψ or R_0 is continuous at $r=R$

② δ function potential; ψ' is discontinuous

$\nabla^2\psi = \frac{2mV}{\hbar^2}\psi - \frac{2mE}{\hbar^2}\psi$ integrate from $R-\epsilon$ to $R+\epsilon$; energy term vanishes

$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}$ (r dependence)

$$\int dr\,\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \int r^2\frac{2mV\psi}{\hbar^2}\,dr$$ r^2 terms cancel out

$$\left.\frac{dR_{0>}}{dr}\right|_R - \left.\frac{dR_{0<}}{dr}\right|_R = \frac{2mvR}{\hbar^2}R_{0<}(R) = \frac{2mvR}{\hbar^2}\frac{C\sin(kR)}{kR}$$

① $A\sin(kR) - B\cos(kR) = C\sin(kR) \Rightarrow \quad C = A - B\frac{\cos(kR)}{\sin(kR)}$

② $\frac{1}{R}A\cos kR + \frac{1}{R}B\sin kR - \frac{1}{R}C\cos kR = \frac{1}{R}\cdot\frac{2mvR}{\hbar^2 k}C\sin kR$

• want to calculate $\frac{B}{A}$, to find the phase shift, therefore substitute C

$A\cos kR + B\sin kR - A\cos kR + B\frac{\cos^2 kR}{\sin kR} = \frac{2mvR}{\hbar^2 k}A\sin kr - \frac{2mvR}{\hbar^2 k}B\cos kr$

Define $\lambda \equiv \frac{2mvR}{\hbar^2}$

$$B\underbrace{\left(\sin kR + \frac{\cos^2 kR}{\sin kR}\right)}_{\frac{1}{\sin kr}} = \frac{\lambda}{k}A\sin kR - \frac{\lambda}{k}B\cos kR$$

$\left(\frac{1}{\sin kr} + \frac{\lambda}{k}\cos kR\right)B = A\frac{\lambda}{k}\sin kR$

$$\frac{B}{A} = \frac{\frac{\lambda}{k}\sin^2 kR}{1+\frac{\lambda}{k}\sin kR\cos kR}$$

$$R_1 = \frac{1}{kr}(A\sin kr - B\cos kr) = \frac{\sqrt{A^2+B^2}}{kr}\sin(kr+\delta_0), \qquad \delta_0 = \tan^{-1}\left(\frac{-B}{A}\right)$$

$$\delta_0 = -\tan^{-1}\left(\frac{\frac{\lambda}{k}\sin^2 kR}{1+\frac{\lambda}{k}\sin kR\cos kR}\right)$$

$$\boxed{\sigma_0 = \frac{4\pi}{k^2}\sin^2\left[\tan^{-1}\left(\frac{\frac{\lambda}{k}\sin^2 kR}{1+\frac{\lambda}{k}\sin kR\cos kR}\right)\right]}$$

b. σ_0 vanishes when $\sin^2\delta_0 = 0$, or $\delta_0 = n\pi$

$\Rightarrow$ or $\tan^{-1}(\quad) = n\pi = 0$ (only 0 is possible)

$$\Rightarrow \frac{\frac{\lambda}{k}\sin^2 kR}{1+\frac{\lambda}{k}\sin kR\cos kR} = 0$$

σ_0 vanishes for $kR = n\pi$

$k^2 = \frac{n^2\pi^2}{R^2} = \frac{2mE}{\hbar^2} \longrightarrow E = \frac{\hbar^2\pi^2 n^2}{2mR^2}$

(or..., it doesn't "feel" the potential; it is not scattered, hence $\sigma \to 0$)

c. when $E \Rightarrow 0$, $k \to 0$ $\quad \sin kR \to kR$, $\cos kR \to 1$

$$\tan^{-1}\left(\frac{\frac{\lambda}{k}k^2R^2}{1+\lambda R}\right) = \tan^{-1}\left(\frac{\lambda kR^2}{1+\lambda R}\right) \longrightarrow \frac{\lambda kR^2}{1+\lambda R}$$ (as long as $1+\lambda R$ not small or 0)

$$\sin^2\left(\frac{\lambda kR^2}{1+\lambda R}\right) \longrightarrow \frac{\lambda^2k^2R^4}{(1+\lambda R)^2}$$

$$\sigma_0 = \frac{4\pi\lambda^2R^4}{(1+\lambda R)^2} \qquad \lambda = \frac{2mvR}{\hbar^2}$$

d. $\sigma_0 = \frac{4\pi}{k^2}\sin^2\left[\tan^{-1}\left(\frac{\lambda kR^2}{1+\lambda R}\right)\right]$ if $1+\lambda R \to 0$, then $\tan^{-1}(\quad) \to \pm\frac{\pi}{2}$,

$\sin^2 \to 1$, $\sigma_0 \to \frac{1}{k^2} = \infty$

$\sigma_0 = \infty$ if $1 + \frac{2mvR^2}{\hbar^2} = 0$

$$\boxed{V = \frac{-\hbar^2}{2mR^2}}$$ V is deep enough to confine the particle