M05Q.2

For S-wave scattering, we need to match the S-component of the angular momentum

$$\Psi = A\left(\frac{e^{ikr}}{r} - \frac{e^{-ikr+2\delta}}{r}\right) \tag{1}$$

to the solution within the potential zone, where δ is the phase shift of the scattered wave. Here the first term is the incoming wave, while the second is the scattered. This can be rewritten as

$$\Psi = A\,\frac{\sin kr + \delta}{r}\,. \tag{2}$$

Inside the potential zone, the Schrodinger equation reads

$$-\frac{\hbar}{2m}\nabla^2\Psi = E\psi \tag{3}$$

which has solution

$$\Psi = B\,\frac{\sin kr}{r}\,, \tag{4}$$

where the cosine term has been dropped as it diverges at $r = 0$. The wave function must be continuous at R,

$$A\sin kR + \delta = A\sin(kr)\sin(\delta) + A\cos(kr)\sin(\delta) = B\sin kR \tag{5}$$

while, since the potential is a delta function, the derivative of the wave function must obey the boundary condition

$$-Ak\frac{\cos kR+\delta}{R}-A\frac{\sin kR+\delta}{R^2}+Bk\frac{\cos kR}{R}+B\frac{\sin kR+\delta}{R^2} = \frac{2mvR}{\hbar^2}B\frac{\sin kR}{R} \tag{6}$$

or, substituting in the first constraint,

$$-kR\cos(kR+\delta)\sin(kR)+kR\cos(kR)\sin(kR+\delta)=\frac{2mvR^2}{\hbar^2}\sin(kR)\sin(kR+\delta). \tag{7}$$

Rearranging and using the trigonometric identities

$$\cos(a\pm b)=\cos(a)\cos(b)\mp\sin(b)\sin(a) \tag{8}$$
$$\sin(a\pm b)=\sin(a)\cos(b)\pm\sin(b)\cos(a) \tag{9}$$

gives

$$-\sin(\delta)=\alpha\sin(kR)[\sin(kR)\cos(\delta)+\sin(\delta)\cos(kR)]. \tag{10}$$

where

$$\alpha=\frac{2mvR}{k\hbar^2}. \tag{11}$$

Rearranging once again gives

$$(\alpha\sin(kR)\cos(kR)+1)\tan(-\delta)=\alpha\sin^2(kR). \tag{12}$$

And finally, solving for δ gives

$$-\delta=\cot^{-1}\left[\cot(kR)+\frac{1}{\alpha\sin^2 kR}\right]. \tag{13}$$

The scattering cross-section in terms of δ is given by

$$\sigma=\frac{4\pi}{k^2}\sin^2\delta. \tag{14}$$

Drawing a triangle, we can quickly see that the sin of a cotan gives us, in this case,

$$\sigma = \frac{4\pi}{k^2 + \left[k\cot(kR) + \frac{k}{\alpha\sin^2 kR}\right]^2} \tag{15}$$

b) Here, the energies will vanish when $\tan$ goes to infinity, or $\sin^2$ goes to zero. This happens at

$$k = \frac{\pi}{2R} \tag{16}$$

or

$$E = \frac{\hbar^2\pi^2}{8mR^2} \tag{17}$$

c) When $E = 0$,

$$\frac{1}{k}\alpha\sin^2 kR = \frac{2mvR^3}{k^2R^2\hbar^2}\sin^2 kR \to \frac{2mvR^3}{\hbar^2} \tag{18}$$

so

$$\sigma = \frac{4\pi}{\left[\frac{1}{R} + \frac{\hbar^2}{2mvR^3}\right]^2} = \frac{4\pi R^2\beta^2}{(1+\beta)^2} \tag{19}$$

where $\beta = 2mvR/\hbar^2$.

d) Here, we want $\beta = -1$, or

$$v = -\frac{\hbar^2}{2mR} \tag{20}$$