M05M.2 - Suspension Bridge Shape

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Problem: *Determine the equilibrium shape $y(x)$ of the suspension cable in a bridge shown in the figure. The length of the roadbed is L and the mass M is so large that you can neglect the weight of the suspension cable and the and the vertical supporting cables. The lengths of the vertical cables are adjusted so there is no shear stress in the roadway. You can also assume that the vertical cables are close enough together that y can be approximated as an analytic function of x.*

Solution: The bridge has a constant mass per unit length that is supported by the cables: $\Delta m = \frac{M}{L}\Delta x$. So the weight supported by each section of suspension cables separated by Δx is

$$w\Delta x = \Delta mg = \frac{Mg\Delta x}{L} \tag{1}$$

To find the shape $y(x)$, consider the horizontal and vertical tension balance, as in equilibrium there is no acceleration. Horizontal tension must equal tension at the base of the rope arc; vertical tension must balance the supported weight of the bridge:

$$T\cos\theta = T_{y=0} \qquad\qquad T\sin\theta = w\Delta x = \frac{Mg\Delta x}{L} \tag{2}$$

Note: Δx (or just call it x now) is a small section of the bridge supported between the cables - it is not the cable length. This is what allows for direct integration! The ratio of the horizontal and vertical tensions is simply $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{T\sin\theta}{T\cos\theta} = \frac{w\Delta x}{T_{y=0}} \to \frac{wx}{T_{y=0}} = \frac{Mgx}{LT_{y=0}} \tag{3}$$

$$\boxed{y(x) = \frac{Mgx^2}{2LT_{y=0}} + C} \tag{4}$$

Thus, in equilibrium, the suspension cable falls in a parabolic shape to support the bridge.

Notes: This initially sounds similar to the catenary problem, which is the shape of a rope falling under its own weight. However, this problem is asking to consider supporting the weight of the bridge with the cable - not its own weight. I found understanding the catenary problem helpful to understanding the given problem. The catenary solution is sketched below (essentially solution to J00M.1):

Tension will balance the gravitational force, so consider the vertical component of it, $T\sin\theta$. Writing out all components of force (and using $\Delta m = \rho l$ since is mass is dependent on the length of rope in a horizontal section - not $\Delta m = \rho\Delta x$, and this difference is why you cannot directly integrate for the catenary problem):

$$T_{y=h} = (T\cos\theta, T\sin\theta) \tag{5}$$

$$T_{y=0} = (-T_{y=0}, 0) \tag{6}$$

$$mg = -\rho lg = (0, -\rho lg) \tag{7}$$

which must sum to zero in equilibrium since there is no acceleration.

$$T_{y=0} = T\cos\theta \qquad\qquad T\sin\theta = \rho lg \tag{8}$$

Note that the vertical component of the tension is length dependent. Then

$$\frac{dy}{dx} = \frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{\rho l g}{T_{y=0}} \tag{9}$$

And the arc length is $\frac{dl}{dx}$:

$$\frac{dl}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{dx}\sqrt{dx^2 + dy^2} = \frac{\rho g}{T_{y=0}}\sqrt{\frac{T_{y=0}^2}{\rho^2 g^2} + l^2} \tag{10}$$

$$\frac{dx}{dl} = \frac{1}{\frac{dl}{dx}} = \frac{T_{y=0}}{\rho g\sqrt{\frac{T_{y=0}^2}{\rho^2 g^2} + l^2}} \tag{11}$$

$$\frac{dy}{dl} = \frac{\frac{dy}{dx}}{\frac{dl}{dx}} = \frac{l}{\sqrt{\frac{T_{y=0}^2}{\rho^2 g^2} + l^2}} \tag{12}$$

And now $\frac{dy}{dl}$ can be integrated:

$$dy = \frac{l}{\sqrt{\frac{T_{y=0}^2}{\rho^2 g^2} + l^2}}dl \tag{13}$$

$$y = \sqrt{\frac{T_{y=0}^2}{\rho^2 g^2} + l^2} \tag{14}$$

An expression for $x(l)$ can also be found:

$$dx = \frac{T_{y=0}}{\rho g\sqrt{\frac{T_{y=0}^2}{\rho^2 g^2} + l^2}}dl \tag{15}$$

$$x = \frac{T_{y=0}}{\rho g}\sinh^{-1}\left(\frac{l\rho g}{T_{y=0}}\right) \tag{16}$$

Substituting $l(x)$ into $y(l)$, this gives the solution to the catenary problem:

$$\boxed{y(x) = \frac{T_{y=0}}{\rho g}\cosh\left(\frac{x\rho g}{T_{y=0}}\right)} \tag{17}$$