1 May 2005, Mechanics, Problem 1

1.1 (a)

There has been a lot of discussion on how to solve this problem. All of us have been trying to set up the right coordinate system, the correct difference of positions, etc, but every time there seems to be something missing. The solution I've liked the most so far is the one I'm presenting. We work in the coordinate system of the base of the piston. This coordinate system is accelerated when the floor moves, so it experiences a "non-inertial force". If x is the coordinate of the displacement of the experiment as seen in the reference frame of the base of the piston, the equation of motion is:

$$M\ddot{x} = PA - Mg + F_{damp} + F_{non-inert}$$

If the floor oscillates as $Be^{i\omega t}$, then the reference frame is accelerated with $a = -\omega^2 Be^{i\omega t}$, and so:

$$F_{non-inert} = -Ma = M\omega^2 Be^{i\omega t}$$

$$F_{damp} = -M\dot{x}/\tau$$

We will find out what P is both for the isothermal and for the adiabatic case.

(i) Adiabatic

$$PV^\gamma = P_0 V_0^\gamma = MgV_0^\gamma/A$$

where the second equality comes from the equilibrium condition $P_0 A = Mg$. After solving for P, and plugging in $V = V_0 + Ax$, where $Ax << V_0$, we get:

$$P = Mg\left(1 - \frac{\gamma Ax}{V_0}\right)/A$$

$$M\ddot{x} = -Mg\frac{\gamma Ax}{V_0} - M\dot{x}/\tau + M\omega^2 Be^{i\omega t}$$

When $\tau \to \infty$, the resonance frequency is the natural frequency, and the natural frequency can be found when $B = 0$. We then find:

$$\omega_0 = \sqrt{\frac{g\gamma A}{V_0}} \qquad (1)$$

(ii) Isothermal

$$PV = NkT = P_0 V_0 = MgV_0/A$$

Solve for P by putting in the change in volume and making a small-x approximation:

$$P = Mg\left(1 - \frac{Ax}{V_0}\right)/A$$

$$M\ddot{x} = -Mg\frac{Ax}{V_0} - M\dot{x}/\tau + M\omega^2 Be^{i\omega t}$$

As before, $\tau \to \infty$ and $B = 0$ give:

$$\omega_0 = \sqrt{\frac{gA}{V_0}} \tag{2}$$

As we can see, the two solutions differ only by a factor of $\sqrt{\gamma}$, so the key aspects of the solution will be the same if we make either assumption. However, we should decide on one of them. In the long run, the system will be isothermal, because the air around it acts as a heat bath. However, for fast oscillations, the system has no time to equilibrate and so it will not release or receive energy from the surroundings. Thus, we will assume the processes are adiabatic.

1.2 (b)

Plug in a solution $x = Ce^{i\omega t}$, and obtain:

$$C(-\omega^2 + \omega_0^2 + i\omega/\tau) = B\omega^2$$

Now, the transmission coefficient is the ratio of the oscillation amplitude at experiment level to the oscillation amplitude at floor level. Bear in mind that x is the position of the experiment relative to the base of the piston, so in fact the total amplitude of oscillation at experiment level is that plus the floor oscillation. Therefore,

$$T = \frac{C+B}{B} = \frac{\omega_0^2 + i\omega/\tau}{-\omega^2 + \omega_0^2 + i\omega/\tau} = \sqrt{\frac{\omega_0^4 + \omega^2\tau^2}{(\omega_0^2 - \omega^2)^2 + \omega^2/\tau^2}}e^{i\phi} \tag{3}$$

where

$$\phi = \frac{-\tau\omega^3}{\tau^2\omega_0^2(\omega_0^2 - \omega^2) + \omega^2}$$

Noticing that the absolute square of the transmission coefficient is 1 at $\omega = 0$, greater than 1 at $\omega = \omega_0$, and 0 as $\omega \to \infty$, we can guess that the transmission coefficient is smallest at large frequencies.

1.3 (c)

There's two ways to see this part:

(i) calculate $|T|$ for $\omega = \omega_0$:

$$|T| = \sqrt{1 + \tau^2\omega_0^2}$$

This quantity is smallest when τ and ω_0 are smallest.

(ii) realize that to minimize oscillations, you want the system to be critically damped, so that the solution to the equation of motion is exponentially decaying:

$$x = De^{-kt}$$

Plugging that in with B=0, we get:

$$k = \frac{-1/\tau \pm \sqrt{1/\tau^2 - 4\omega_0^2}}{2}$$

The root will be real only when:

$$\frac{1}{\tau^2} > 4\frac{g\gamma A}{V_0} \tag{4}$$

Notice that in either case the conclusion is the same: we want to minimize the time constant as much as possible, minimize the area of the cylinder, and maximize the volume.